Topics in Algebra - Singularities in Positive Characteristic

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Contents

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Problem Sets

Problem Set 1

Throughout, R is a fixed noetherian ring of characteristic $p > 0$.

1. Prove the Frobenius is injective if and only if R is reduced.

2. Prove when the Frobenius map is flat, for ideals $\mathfrak{a}$ and $\mathfrak{b}$, one has
$$(\mathfrak{a} : \mathfrak{b})^{[q]} = \left(\mathfrak{a}^{[q]} : \mathfrak{b}^{[q]}\right).$$

3. Prove for a multiplicatively closed set W, $W^{-1}F^e_* R \cong F^e_*(W^{-1}R)$.

4. Prove the Frobenius map induces the identity map on $\operatorname{Spec} R$.

5. For any ideal $\mathfrak{a}$, prove the powers $\{\mathfrak{a}^n\}_n$ and the Frobenius powers $\{\mathfrak{a}^{[p^e]}\}_e$ are cofinal.

6. Prove that F_*- is an exact functor on R-**mod**.

7. If $R \to S \to T$ are maps of rings for which T is a flat R-module, and T is a faithfully flat S-module, prove S is a flat R-module.

8. Prove a finitely generated flat module over a local ring is free. (Hint: Note that flat modules are projective, and use Nakayama's Lemma.)

9. Prove any F-finite local ring $(R, \mathfrak{m}, k)$ with perfect residue field has F_*R minimally generated by $\dim_k \left(R/\mathfrak{m}^{[p]} \right)$ generators. What changes if k is not perfect?

Problem Set 2

Throughout, R is a fixed noetherian ring of characteristic $p > 0$.

1. Prove an F-split ring is reduced.

2. Prove any localization of an F-split ring remains F-split.

3. Decide in which characteristics $f = x^3 + y^3 + z^3 \subseteq \mathbf{F}_p[x, y, z]_{\mathfrak{m}}$ defines an F-split hypersurface where $\mathfrak{m} = (x, y, z)$.

4. Show the ideal I_2 of 2-minors of a 2×3-matrix is F-split.

5. Fix a ring R and $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$. Show when $\mathfrak{a}$ and $\mathfrak{b}$ are φ-compatible, $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}\cap\mathfrak{b}$, $\sqrt{\mathfrak{a}}$, and $(\mathfrak{c}:\mathfrak{a})$ for any ideal $\mathfrak{c}$ remain φ-compatible.

6. Prove when R is F-split and $\mathfrak{p}$ is a minimal prime, $R/\mathfrak{p}$ is also F-split.

7. For $\mathfrak{p}$ prime in a regular ring R, prove $\mathfrak{p}^{[p^e]}$ is $\mathfrak{p}$-primary.

8. When R is regular and I is a radical ideal, prove $\mathrm{Ass}\left(R/I\right) = \mathrm{Ass}\left(R/I^{[q]}\right)$ for any $q = p^e$.

Problem Set 3

Throughout, R is a fixed noetherian ring of characteristic $p > 0$.

1. Prove the functor $\Gamma_\mathfrak{m}$ is left exact.

2. Fix an extension of rings $R \to S$, not necessarily flat. Given a complex $M^\bullet$, determine a natural map $h^n(M^\bullet)\otimes_R S \to h^n(M^\bullet \otimes_R S)$ and prove it is S-linear.

3. Prove any permutation of a regular sequence on a finitely generated R-module remains a regular sequence when R is local.

4. Fix a local ring $(R,\mathfrak{m})$. A regular sequence of length 1 is called a **regular element**, i.e., a non-zero divisor non-unit x. Prove that being Cohen-Macaulay 'deforms,' i.e., if R/x is Cohen-Macaulay, then R is Cohen-Macaulay.

5. For a ring R and $\mathfrak{a} = (f_1, ..., f_s)$, verify a class $\left[\dfrac{g}{{f_1}^a \cdots {f_s}^a}\right] = 0$ in $H^s_\mathfrak{a}(R)$ if and only if there is a non-negative integer k so that $g(f_1\cdots f_s)^k \in \left({f_1}^{a+k}, ..., {f_s}^{a+k}\right)$.

6. Prove that in a local ring $(R,\mathfrak{m})$, one has isomorphisms as F_*R-modules $H^i_\mathfrak{m}(R)\otimes_R F_*R \cong H^i_\mathfrak{m}(F_*R) \cong F_*H^i_\mathfrak{m}(R)$. (Hint: Consider the Čech complex.) More generally, prove when $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a finite local extension, that is, $\mathfrak{m}S = \mathfrak{n}$, and S is a finitely generated R-module, $H^i_\mathfrak{n}(R)\otimes_R S \cong H^i_\mathfrak{m}(S)$.

7. Verify for a ring R, any $R\{F\}$-module W, and $y \in W$, the submodule $\mathrm{span}_R\{\rho(y), \rho^2(y), ...\}$ is F-stable.

8. Prove if W and W' are $R\{F\}$-modules, if W is an $R\{F\}$-submodule of W' and W' is anti-nilpotent, then W is also. Furthermore, if $W \to W'$ is surjective and W is anti-nilpotent, then W' is too. Use this to show in a short exact sequence of $R\{F\}$-modules, if any two are anti-nilpotent, then the third one is too.

9. Suppose $(R, \mathfrak{m}_R)$ is a regular local ring with any characteristic, and $R \subseteq S$ is a local extension; i.e., $(S, \mathfrak{m}_S)$ is local and $\mathfrak{m}_R S = \mathfrak{m}_S$. Prove when S is a finitely generated R-module, then S is Cohen-Macaulay if and only if it is free as an R-module. (Hint: First note that $\operatorname{depth} S$ is the same if we think of it as an R-module or as an S-module. Recall the Auslander-Buchsbaum theorem.)

10. Call an $R\{F\}$-module (M, ρ) **nilpotent** provided for each $m \in M$, there is e so that $\rho^e(m) = 0$. Prove in a short exact sequence $0 \to M \to N \to P \to 0$ of $R\{F\}$-modules, N is nilpotent if and only if M and P are.

Problem Set 4

Throughout, R is a fixed noetherian ring of characteristic $p > 0$.

1. Prove any F-injective ring is reduced.

2. For a surjective element x, for each $\ell > 0$ and $j \geq \ell$, the maps $H^i_{\mathfrak{m}}\left({}^{R}\!/\!_{x^\ell R}\right) \to H^i_{\mathfrak{m}}\left({}^{R}\!/\!_{x^j R}\right)$ are injective.

3. Prove that a regular element x is a surjective element if and only if the multiplication map $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)$ is surjective for all i.

4. Recall for an $R\{F\}$-module (M, ρ), set $0^\rho_M = \{m \in M \mid \text{ there exists } e \text{ such that } \rho^e(m) = 0\}$. Prove for a short exact sequence $0 \to A \to B \to C \to 0$, $0^\rho_B = B$ if and only if $0^\rho_A = A$ and $0^\rho_C = C$.

5. Suppose $L \to M \to N$ is an exact sequence of $R\{F\}$-modules. If L is anti-nilpotent and ρ_N is injective, prove that ρ_M is injective.

6. Prove a gluing theorem for anti-nilpotent.

Problem Set 5

Throughout, R is a fixed noetherian ring of characteristic $p > 0$.

1. For $(R, \mathfrak{m}, k)$ an Artin local ring and $E = E_R(k)$, for any finitely generated R-module M, prove the natural map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)$ is an isomorphism.

2. For $(R, \mathfrak{m}, k)$ a complete local ring, prove, for an R-module M, the map $R \to M$ splits if and only if $E \cong E \otimes_R R \to E \otimes_R M$ is injective.

3. Prove when $S \to R$ is a map of rings for which R is a finite S-module and $\omega^\bullet_S$ is a dualizing complex for S, that $\mathbf{R}\operatorname{Hom}_S(R, \omega^\bullet_S)$ is a dualizing complex for R.

4. Prove when $S \to R$ is a split map of rings, if R has FFRT, then so does S.

5. If R is a ring with $0 \to S \to R \to {}^{R}\!/_{\mathfrak{a}} \to 0$ a short exact sequence, provided both S and ${}^{R}\!/_{\mathfrak{a}}$ have FFRT, must R also have FFRT?

Semester 1

Some course notes are available at

1. Utah, K. Schwede (2010, 2017),
2. Michgan, K. Smith (2018), and
3. papers in the literature.

1.1 Overview

Fix a space X and a point $x_0 \in X$. We have an associated local ring $\mathcal{O}_{X,x_0} = (R, \mathfrak{m}, k)$ where $k = R/\mathfrak{m}$. We always assume $k \subseteq R$. Rings are commutative and unital unless otherwise mentioned.

Definition 1.1.1 (sheaf)**.** A **sheaf** $\mathcal{F}$ is a contravariant functor $\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathcal{C}$ satisfying the gluing condition; i.e., given a open cover $U = \bigcup U_i$,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

Definition 1.1.2 (spectrum)**.** Given a ring R, the **(prime) spectrum** of R, denoted $X = \operatorname{Spec} R$, is the set $\{\mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ is a prime ideal}\}$ endowed with the Zariski topology and structure sheaf $\mathcal{O}_X$, making $(X, \mathcal{O}_X)$ a locally ringed space. The Zariski topology has closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$. The Zariski topology has a basis of standard open sets, denoted $\mathcal{B} = \{D_f\}$, where $D_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in R, f \notin \mathfrak{p}\}$. The structure sheaf $\mathcal{O}_X$ is the unique sheaf of rings for which $\mathcal{O}_X(X) = R$ and $\mathcal{O}_X(D_f) = R_f$ for all standard opens D_f. The stalk of the structure sheaf at a point $x_0 \in X$ is the local ring $\mathcal{O}_{X,x_0} = \varinjlim_{x \in U} \mathcal{O}_X(U)$.

Definition 1.1.3 (scheme)**.** A locally ringed space which is isomorphic to $\operatorname{Spec} R$ for some R is called an affine scheme. A **scheme** is a locally ringed space which admits a covering by open sets U_i such that each U_i is an affine scheme.

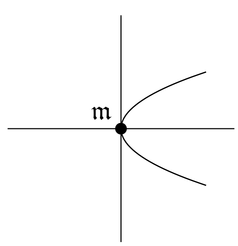
Definition 1.1.4 (Krull dimension)**.** We define the **Krull dimension** of a ring R to be

$$\dim R = \sup_n \{\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \mid \mathfrak{p}_i \subseteq R \text{ is a prime ideal}\}.$$

Definition 1.1.5 (regular ring)**.** A local ring $(R, \mathfrak{m}, k)$ is **regular** if R is noetherian, $\mathfrak{m} = (f_1, ..., f_n)$ where n is minimal, and $\dim R = n$. A ring R is **regular** if $\left(R_{\mathfrak{p}}, \mathfrak{p}, R_{\mathfrak{p}}/\mathfrak{p}\right)$ is regular for every prime ideal $\mathfrak{p} \subseteq R$.

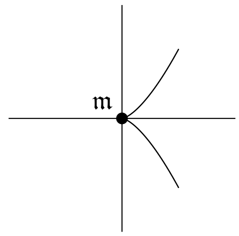
Definition 1.1.6 (nonsingular point)**.** A point $x_0 \in X$ is a **nonsingular point** (or a **manifold point**) if and only if $\mathcal{O}_{X,x_0}$ is regular.

Example 1.1.7. Let $X = \operatorname{Spec} \mathbf{C}[x,y]/(y^2 - x)$, $\mathfrak{m} = (x, y)$.



$\mathcal{O}_{X,x_0} = \mathbf{C}[x,y]_{\mathfrak{m}}/(y^2 - x)$ is a regular ring, so x_0 is a regular point.

Example 1.1.8. Let $X = \operatorname{Spec} \mathbf{C}[x,y]/(y^2 - x^3)$, $\mathfrak{m} = (x, y)$.



$\mathcal{O}_{X,x_0} = {\mathbf{C}[x,y]_{\mathfrak{m}}}/{(y^2 - x^3)}$ is not a regular ring, so x_0 is not a regular point.

Remark 1.1.9. The study of singularities over $\mathbf{C}$ has many tools:

1. small open balls (i.e., local methods),
2. GAGA theorems, allowing us to use analytic approaches (i.e., integration),
3. resolution of singularities and the Minimal Model Program,
4. etc.

None of this is available if we replace $\mathbf{C}$ by $\mathbf{F}_p$. We do, however, gain a new tool:

Definition 1.1.10 (Frobenius)**.** Let $(R, \mathfrak{m})$ be a local ring, with $p = 0$. The p-power map $F : R \to R$, $f \mapsto f^p$, is a ring homomorphism; i.e., $(f+g)^p = f^p + g^p$. F is called the **Frobenius**.

Definition 1.1.11 (F_*R)**.** Set, for a characteristic $p > 0$ ring R, a new module

$$F_*R = \{F_*r \mid r \in R\}$$

and identify $F_*R \cong R$ as a group. That is, $F_*r + F_*s = F_*(r+s)$. Let the R-module structure be given by

$$sF_*r = F_*(s^p r).$$

Remark 1.1.12. F_*M makes sense for any R-module M. The natural map $R \to F_*R$ sending $1 \mapsto F_*1$ is identified with the Frobenius; i.e., $R \to F_*R$ sends $r \mapsto rF_*1 = F_*(r^p)$.

Remark 1.1.13. For $R = \mathbf{F}_p(x_1, \ldots)$, F_*R is not a finitely generated R-module. (Though R is; it is a field, hence a 1-dimensional vector space over itself.)

We're most interested in the situation where F_*R is a finitely generated R-module.

Definition 1.1.14 (F-finite)**.** We call such rings (that is, rings where F_*R is a finitely generated R-module) **F-finite**.

Remark 1.1.15. Being F-finite is a robust notion; F-finite rings are closed under localization, taking polynomial algebras, completion, etc.

Definition 1.1.16 (perfect field)**.** A field k of characteristic p is **perfect** if every element of k is a p-th power; i.e., $k = k^p$.

Remark 1.1.17. Perfect fields $k = k^p$ are always F-finite.

Remark 1.1.18. A general perspective to have in mind is $R \cong {k[x_1, \ldots, x_n]}/{\mathfrak{a}}$ (of finite type), or a localization of a finite type ring (essentially of finite type), where k is perfect of characteristic $p > 0$. All such rings are F-finite.

We can also construct new ideals using the Frobenius.

Definition 1.1.19 (Frobenius power)**.** Let R be a ring and $\mathfrak{a} \subseteq R$, $\mathfrak{a} = (f_1, \ldots, f_m)$, be an ideal. We have

$$\mathfrak{a}F_*R = F_*\mathfrak{a}^{[p]},$$

where $\mathfrak{a}^{[p]} = ({f_1}^p, \ldots, {f_m}^p)$. We call $\mathfrak{a}^{[p]}$ the **Frobenius power** of $\mathfrak{a}$.

Remark 1.1.20. Note $\mathfrak{a}^{[p]} \subseteq \mathfrak{a}^p$, but is often much smaller.

Example 1.1.21. Let $R = k[x]$ with $k = k^p$. Then F_*R is free of rank p, with basis $\{F_*1, F_*x, F_*x^2, ..., F_*x^{p-1}\}$.

Remark 1.1.22. A key feature of the Frobenius $F : R \to R$ is that we can iterate; $F^e : R \to R$ sends $f \mapsto f^{p^e}$. We also have $F^e_*R = \{F^e_*r \mid r \in R\}$ and $sF^e_*r = F^e_*\left(s^{p^e}r\right)$. Furthermore, $\mathfrak{a}^{[p^e]} = \left({f_1}^{p^e}, ..., {f_m}^{p^e}\right)$ for $\mathfrak{a} = (f_1, ..., f_m) \subseteq R$.

Example 1.1.23. Let $R = k[x_1, ..., x_d]$. Then F^e_*R is free with basis $\{F^e_*x_1^{t_1}, ..., F^e_*x_d^{t_d}\}$ with $t_i < p^e$. So, $\operatorname{rank}(F^e_*R) = p^{ed}$.

Theorem 1.1.24 (Kunz). *For a local ring R of dimension d, the following are equivalent:*

1. *R is regular,*
2. *F is flat, and*
3. *F_*R is free of rank p^d.*

1.2 Review: Dimension Theory

Fix any ring R.

Definition 1.2.1 (catenary)**.** Call a ring R **catenary** if for any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ in $\operatorname{Spec} R$, all maximal chains $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{q}$ have the same length.

Definition 1.2.2 (height)**.** For $\mathfrak{p} \in \operatorname{Spec} R$, define the **height** of $\mathfrak{p}$, $\operatorname{ht}\mathfrak{p}$, to be $\sup_n\{\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}$. For $\mathfrak{a} \subseteq R$ any ideal, set $\operatorname{ht}\mathfrak{a} = \min_{\mathfrak{p} \supsetneq \mathfrak{a}} \operatorname{ht}\mathfrak{p}$.

Definition 1.2.3 (Krull dimension 2)**.** The **Krull dimension** of a ring R is

$$\dim R = \sup_{\mathfrak{m} \text{ maximal}} \operatorname{ht}\mathfrak{m}.$$

Remark 1.2.4. For rings essentially of finite type, one has $\dim R = \dim\left({}^{R}\!/_{\mathfrak{p}}\right) + \operatorname{ht}\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 1.2.5 (radical)**.** For an ideal $\mathfrak{a} \subseteq R$, the **radical** of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} = \{r \in R \mid r^n \in \mathfrak{a} \text{ for some } n \in \mathbf{N}\}.$$

Definition 1.2.6 (system of parameters)**.** For a local ring $(R, \mathfrak{m})$, we call a sequence $x_1, ..., x_d$ a **system of parameters** if $\sqrt{(x_1, ..., x_d)} = \mathfrak{m}$ and d is minimal.

Definition 1.2.7 (regular ring 2)**.** A local ring $(R, \mathfrak{m})$ is **regular** provided $\mathfrak{m}$ is generated by $\dim R$ elements.

Remark 1.2.8. A natural question arises: how can we actually find the minimal number of generators of $\mathfrak{m}$ in order to check regularity?

Lemma 1.2.9 (Nakayama's Lemma)**.** *Let $(R, \mathfrak{m})$ be local, and let M be a finitely generated R-module. If ${}^{M}\!/_{\mathfrak{m}M} = 0$, then $M = 0$.*

Remark 1.2.10. This forces of a lift of a generating set for ${}^{M}\!/_{\mathfrak{m}M}$ to be a generating set for M. Indeed, suppose $\overline{\beta}$ generates ${}^{M}\!/_{\mathfrak{m}M}$. Set $N = \langle \beta \rangle$ where β is a lift of $\overline{\beta}$. Then

$$\left({}^{M}\!/_{N}\right)\Big/_{\mathfrak{m}\left({}^{M}\!/_{N}\right)} = 0,$$

so by **Lemma 1.2.9 [Nakayama's Lemma]**, ${}^{M}\!/_{N} = 0$; i.e., $M = N$. Apply this to $M = {}^{R}\!/_{\mathfrak{m}}$. We see that the minimal number of generators of $\mathfrak{m}$ is equal to $\dim_k\left({}^{\mathfrak{m}}\!/_{\mathfrak{m}^2}\right)$. In other words, the embedding dimension is the dimension of the Zariski cotangent space!

1.3 Kunz's Theorem

Recall **Theorem 1.1.24 [Kunz]**. A key tool to prove this is completion of rings.

Definition 1.3.1 (completion)**.** For any local ring $(R, \mathfrak{m})$, the **($\mathfrak{m}$-adic) completion** is

$$\widehat{R} = \varprojlim_n R/\mathfrak{m}^n,$$

a new ring whose elements are $(r_i)_{i\in\mathbf{N}}$ with $r_i \in R/\mathfrak{m}^i$ and $r_j \equiv r_i \bmod \mathfrak{m}^j$ if $j \leq i$.

Definition 1.3.2 (complete)**.** There is a natural map $R \xrightarrow{\varphi} \widehat{R}$ where $r \mapsto (..., r, r, r)$. We call R **complete** if φ is an isomorphism.

Definition 1.3.3 (complete module)**.** One can do a similar construction for an R-module N. Let

$$\widehat{N} = \varprojlim_n N/\mathfrak{m}^n N$$

be the $\mathfrak{m}$-adic completion, and call N **complete** if $N \to \widehat{N}$ is an isomorphism.

Example 1.3.4. Let $R = k[x]_{(x)}$ with $\mathfrak{m} = (x)$. An element of $\widehat{R}$ is a family of polynomials

$$(f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, f_3 + \mathfrak{m}^3, ...) = (c_0, c_0 + c_1 x, c_0 + c_1 x + c_2 x^2, ...).$$

One may identify $\widehat{R} \cong k[\![x]\!]$.

Example 1.3.5. Let $R = \mathbf{Z}_{(p)}$ with p prime; that is, R is the localization at (p), $\mathbf{Z}\left[\frac{1}{p}\right]$. Then observe that

$$\widehat{R} = \varprojlim_n \mathbf{Z}_{(p)}/p^n\mathbf{Z}_{(p)} \cong \varprojlim_n \mathbf{Z}/p^n\mathbf{Z} = \mathbf{Z}_p$$

which are the p-adic numbers.

Remark 1.3.6. Many properties between R and $\widehat{R}$ are shared. In fact, the natural map $R \to \widehat{R}$ is faithfully flat.

Theorem 1.3.7 (Cohen Structure Theorem)**.** *If $(R, \mathfrak{m}, k)$ is a noetherian local ring of finite dimension and $k \subseteq R$, then $\widehat{R} \cong k[\![x_1, ..., x_d]\!]/\mathfrak{a}$. That is, the following are equivalent:*

1. *R is regular,*
2. *$\widehat{R}$ is regular, and*
3. *$\widehat{R} \cong k[\![x_1, ..., x_d]\!]$.*

Definition 1.3.8 (completion of a module)**.** We can also take the **completion** of an R-module M. That is, if M is an R-module with $(R, \mathfrak{m}, k)$ a local ring, then

$$\widehat{M} = \varprojlim_n M/\mathfrak{m}^n M.$$

Remark 1.3.9. To prove **Theorem 1.1.24 [Kunz]**, it is enough to show 1 holds if and only if 2 does; that is, we can show R is regular if and only if F is flat. The freeness of F_*R implies and is implied by the other two. Indeed, see **Problem Set 1 #8**.

Lemma 1.3.10. *For a local ring R, $\widehat{F_*R} \cong F_*\widehat{R}$.*

Proof. Note that the powers $\mathfrak{m}^n$ and the Frobenius powers $\mathfrak{m}^{[p^e]}$ are cofinal. (See **Problem Set 1 #5**.) This gives an isomorphism $\widehat{R} \cong \varprojlim_n R/(\mathfrak{m}^n)^{[p]}$. Thus

$$\widehat{F_*R} = \varprojlim_n F_*R/\mathfrak{m}^n F_*R \cong \varprojlim_n F_*\left(R/(\mathfrak{m}^n)^{[p]}\right).$$

As F_*- is exact (**Problem Set 1 #6**), we have

$$\varprojlim_n F_* \left({}^{R}\!\diagup_{(\mathfrak{m}^n)^{[p]}} \right) \cong F_* \varprojlim_n {}^{R}\!\diagup_{(\mathfrak{m}^n)^{[p]}} \cong F_* \widehat{R},$$

as desired. $\square$

Remark 1.3.11. Observe that by **Remark 1.3.6** and **Lemma 1.3.10**, the natural map $F_* R \to \widehat{F_* R} \cong F_* \widehat{R}$ is faithfully flat. This will help show that 1 implies 2 in **Theorem 1.1.24 [Kunz]**, the easy direction. Consider the diagram

$$\begin{tikzcd}
\widehat{R} \arrow[r] & F_* \widehat{R} \\
R \arrow[r] \arrow[u] & F_* \arrow[u]
\end{tikzcd}$$

where the vertical maps are faithfully flat. As R is regular, by **Theorem 1.3.7 [Cohen Structure Theorem]**, $\widehat{R} \cong k[\![x_1, ..., x_d]\!]$, which is a domain. Hence, we can identify $F_* \widehat{R}$ with $\left(\widehat{R}\right)^{\frac{1}{p}}$, the module of p^{th} roots of $\widehat{R}$ in any fixed algebraic closure of its fraction field. The Frobenius is identified with the natural inclusion

$$\begin{aligned} \widehat{R} &\to \left(\widehat{R}\right)^{\frac{1}{p}} \\ f &\mapsto \left(f^{\frac{1}{p}}\right)^p. \end{aligned}$$

So the map $\widehat{R} \to F_* \widehat{R}$ in the diagram above can be identified with

$$k[\![x_1, ..., x_d]\!] \subseteq k \left[\!\!\left[{x_1}^{\frac{1}{p}}, ..., {x_d}^{\frac{1}{p}} \right]\!\!\right] \subseteq k^{\frac{1}{p}} \left[\!\!\left[{x_1}^{\frac{1}{p}}, ..., {x_d}^{\frac{1}{p}} \right]\!\!\right],$$

where the first inclusion is a free extension, hence flat, and the second inclusion is a base change along a finite field extension, hence also flat. Thus, $\widehat{R} \to \left(\widehat{R}\right)^{\frac{1}{p}} \cong F_* \widehat{R}$ is flat.

$$\begin{tikzcd}
\widehat{R} \arrow[r, "\text{flat}"] & F_* \widehat{R} \\
R \arrow[r] \arrow[u, "\text{faithfully flat}"] & F_* R \arrow[u, "\text{faithfully flat}"']
\end{tikzcd}$$

Remark 1.3.12. As abstract rings, $R \cong F_* R$, and when R is a domain, $F_* R \cong R^{\frac{1}{p}}$. Under these isomorphisms,

$$\begin{tikzcd}
1 \arrow[r, mapsto] & F_* 1 \\[-15pt]
R \arrow[r] \arrow[d, equal] & F_* R \arrow[d, "\wr|"] \\
R \arrow[r, hook] & R^{\frac{1}{p}}
\end{tikzcd}$$

*Proof of **Theorem 1.1.24 [Kunz]**, Part 1.* Let R be regular. By **Remark 1.3.11**, $R \to F_* \widehat{R}$ is flat. By **Problem Set 1 #7**, $R \to F_* R$ is flat. Hence, F is flat, and 1 implies 2. $\square$

Remark 1.3.13. The converse (2 implies 1) requires more. One can still use the **Theorem 1.3.7 [Cohen Structure Theorem]**; a sketch follows. Assume F is flat and $\widehat{R} \cong {}^{k[\![x_1, ..., x_d]\!]}\!\diagup_{\mathfrak{a}}$. The goal is to show that $\mathfrak{a} = 0$. To do this, it is enough to show that

$$\dim_k \left({}^{\widehat{R}}\!\diagup_{\mathfrak{m}^{[p^e]}} \right) = p^{ed},$$

where $d = \dim R$. One way to show this is due to Lech. By flatness of F,

$$F_*\left(\mathfrak{m}^{[p^e]} \big/ \left(\mathfrak{m}^{[p^e]}\right)^2\right) \cong \left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_R F_*R,$$

and $\mathfrak{m}/\mathfrak{m}^2$ is a free $R/\mathfrak{m}$-module. This forces the generators ${x_1}^{p^e}, ..., {x_d}^{p^e}$ of $\mathfrak{m}^{[p^e]}$ to be "Lech independent;" i.e., if $\sum f_i {x_i}^{p^e} = 0$, then $f_i \in \mathfrak{m}^{[p^e]}$. Then one can use induction to prove

$$\dim_k \left(R/({x_1}^{a_1}, ..., {x_d}^{a_d})\right) = \prod a_i$$

for any $(a_1, ..., a_d) \in \mathbf{N}^d$.

Remark 1.3.14. None of the above can work outside characteristic $p > 0$. So, another approach uses derived categories.

1.3.1 Derived Categories

Definition 1.3.15 (derived category)**.** For a ring R, the **derived category of R-mod**, $D(R)$, has objects consisting of chain complexes

$$M_\bullet : \qquad \cdots \to M_3 \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \to \cdots$$

with $d_i d_{i+1} = 0$. We call M_i the degree i part of $M_\bullet$ and $\operatorname{supp} M_\bullet = \{i \mid M_i \neq 0\}$ the support. The arrows in $D(R)$ are yet to come.

Example 1.3.16. Every R-module M gives a complex denoted $[M]$ which is $0 \to M \to 0$.

Remark 1.3.17. We can shift complexes; $(M_\bullet[n])_i = M_{i+n}$.

Example 1.3.18. If R is noetherian and M is finitely generated, then there is a free resolution of M

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Definition 1.3.19 (acyclic)**.** Call a complex $M_\bullet \in \operatorname{obj}(D(R))$ **acyclic** if it is exact. In **Example 1.3.18** above, $F_\bullet \to M \to 0$ is acyclic.

Remark 1.3.20. The derived category offers a way to "replace" $[M]$ with $F_\bullet$. We want $[M]$" $\cong$ "$F_\bullet$.

Definition 1.3.21 (homology)**.** Define $h_n : D(R) \to R$-**mod** to be

$$M_\bullet \mapsto h_n(M_\bullet) = \ker(M_n \to M_{n-1}) \big/ \operatorname{im}(M_{n+1} \to M_n),$$

the **homology** of $M_\bullet$.

Remark 1.3.22. Note $M_\bullet$ is acyclic if and only if $h_n(M_\bullet) = 0$ for all n. Also, note that $h_n(F_\bullet) \cong h_n([M])$.

Definition 1.3.23 (quasi-isomorphism)**.** A basic morphism of complexes $M_\bullet \to N_\bullet$ is a family of maps $M_i \to N_i$ making

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M_{-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & N_{-1} & \longrightarrow & \cdots \end{array}$$

commute. A basic morphism is a **quasi-isomorphism** if the induced map $h_n(M_\bullet) \to h_n(N_\bullet)$ is an isomorphism for all n. We will write $M_\bullet \cong_q N_\bullet$.

☡ **Warning! 1.3.24.** Not all quasi-isomorphisms are invertible!

Remark 1.3.25. Verdie constructed an augmentation to morphisms of complexes so that one may formally invert quasi-isomorphisms. (See: localization of categories.) Precisely:

Definition 1.3.26 (derived category 2)**.** The **maps** in $D(R)$ are (homotopy equivalence classes of) basic morphisms of chain complexes, with quasi-isomorphisms formally inverted. Explicitly, a map in $D(R)$, $M_\bullet \to N_\bullet$, is a diagram (a roof):

$$\begin{tikzcd} & T_\bullet \arrow[dl, "g"'] \arrow[dr, "f"] & \\ M_\bullet & & N_\bullet \end{tikzcd}$$

with g a quasi-isomorphism and f a basic morphism.

Remark 1.3.27. The derived category makes derived functors easier to work with. Recall:

Definition 1.3.28 (left derived functor)**.** For a functor $G : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ which is right exact, its **left derived functors** are

$$\mathbf{L}_n G(M) = h_n(G(F_\bullet))$$

for any free resolution $F_\bullet \to M \to 0$.

Example 1.3.29. Fix $N \in \mathrm{obj}(R\text{-}\mathbf{mod})$. Let $G_N(-) = - \otimes_R N$, and $\mathbf{L}_n G_N(M) = \mathrm{Tor}_n^R(M, N)$.

Remark 1.3.30. Psychologically, replace $[M]$ with $F_\bullet$ and compute $h_n(G(F_\bullet))$. In $D(R)$, there is a functor $\mathbf{L}G : D(R) \to D(R)$ with $M_\bullet \mapsto \mathbf{L}G(M_\bullet)$, and when $M_\bullet = [M]$, $h_n(\mathbf{L}G([M])) = \mathbf{L}_n G(M)$. In fact, $\mathbf{L}G([M]) \cong_q G(F_\bullet)$ for any $F_\bullet \to M \to 0$.

Example 1.3.31. Let $G = G_N(-) = - \otimes_R N$. For any R-module M, $\mathbf{L}G([M]) = M \otimes_R^{\mathbf{L}} N$ is a complex, called the **derived tensor product**, such that $h_n(M \otimes_R^{\mathbf{L}} N) = \mathrm{Tor}_n^R(M, N)$.

Remark 1.3.32. There is a dual notation for complexes; i.e., we could write

$$M^\bullet : \qquad \cdots \to M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots$$

with $d^{i+1}d^i = 0$. In fact, each $M^\bullet$ gives a $M_\bullet$ by $M^n = M_{-n}$. We set

$$h^n(M^\bullet) = {\ker(M^n \to M^{n+1})}\big/{\mathrm{im}(M^{n-1} \to M^n)}.$$

For any left exact functor $F : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$, there is a right derived functor which we denote $\mathbf{R}F : D(R) \to D(R)$, computed via injective resolution; i.e., a quasi-isomorphism $[M] \cong_q E^\bullet$ with $E^\bullet$ acyclic in degree $n > 0$ and E^n injective.

Example 1.3.33. We have $\mathbf{R}\operatorname{Hom}(-, N) : D(R) \to D(R)$. If M is an R-module, then we have $h^n(\mathbf{R}\operatorname{Hom}(M, N)) = \mathrm{Ext}_R^n(M, N)$.

Remark 1.3.34. Derived functors help by carrying a lot of information (every Tor or Ext module) in a compact way (a single complex).

Proposition 1.3.35 (Derived Hom-Tensor Adjunction)**.** *For complexes $M^\bullet$, $N^\bullet$, and $P^\bullet$,*

$$\mathbf{R}\operatorname{Hom}(M^\bullet \otimes_R^{\mathbf{L}} N^\bullet, P^\bullet) \cong_q \mathbf{R}\operatorname{Hom}(M^\bullet, \mathbf{R}\operatorname{Hom}(N^\bullet, P^\bullet)).$$

☡ **Warning! 1.3.36.** $D(R)$ is **not** an abelian category, so we do not have exact sequences of objects in $D(R)$! Verdie, in his thesis, constructed a structure on $D(R)$ that replaces exact sequences. He identified exact triangles. This is a diagram:

$$C^\bullet \to D^\bullet \to F^\bullet \xrightarrow{+1} C^\bullet$$

where $F^\bullet \xrightarrow{+1} C^\bullet$ means a map $F^\bullet \to C^\bullet[+1]$. That is, there is a long exact sequence

$$\cdots \to C^n \to D^n \to F^n \to C^{n+1} \to \cdots .$$

Remark 1.3.37. The following are facts about exact triangles:

1. Any morphism $C^\bullet \to D^\bullet$ can be completed to a triangle. (This uses the mapping cone.)
2. If $C^\bullet \to D^\bullet \to F^\bullet \xrightarrow{+1} C^\bullet$ is a triangle, so too are
$$D^\bullet \to F^\bullet \to C^\bullet[+1] \xrightarrow{+1} D^\bullet$$
and
$$F^\bullet[-1] \to C^\bullet \to D^\bullet \xrightarrow{+1} F^\bullet[-1].$$
3. Given a short exact sequence $0 \to M \to N \to P \to 0$ of R-modules and F a left exact functor, there is an exact triangle $\mathbf{R}F(M) \to \mathbf{R}F(N) \to \mathbf{R}F(P) \xrightarrow{+1} \mathbf{R}F(M)$.

Theorem 1.3.38 (Auslander-Buchsbaum-Serre)**.** *A local ring $(R, \mathfrak{m})$ is regular if and only if it has finite Tor-dimension; i.e., for any two R-modules M and N, $M \otimes_R^{\mathbf{L}} N$ is acyclic in degree $n \geq \dim R$; i.e., $h_n(M \otimes_R^{\mathbf{L}} N) = \mathrm{Tor}_n^R(M, N) = 0$ for $n > \dim R$.*

Remark 1.3.39. Finite Tor-dimension does not persist under quotients. That is, if S has finite Tor-dimension and $R \cong {}^{S}\!/_{\mathfrak{a}}$, then R can fail to have finite Tor-dimension. But we will see a class of rings for which finite Tor-dimension does descend along quotients.

Definition 1.3.40 (perfect ring)**.** A ring R is **perfect** if the Frobenius is an isomorphism.

Example 1.3.41. $\mathbf{F}_p$ is perfect by Fermat's Little Theorem; $a^p = a$ for all $a \in \mathbf{F}_p$.

Example 1.3.42. $\mathbf{F}_p\left[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots\right]$ is perfect.

Theorem 1.3.43 (Bhatt-Scholze)**.** *For R a perfect ring and S and T perfect R-algebras,*
$$S \otimes_R^{\mathbf{L}} T \cong_q S \otimes_R T.$$

Corollary 1.3.44. *If $R \to S$ is a surjection of perfect rings and R has finite Tor-dimension, so too does S.*

Proof. Let M and N be S-modules. We will show that $M \otimes_S^{\mathbf{L}} N \cong_q M \otimes_R^{\mathbf{L}} N$ in $D(R)$. By **Theorem 1.3.43 [Bhatt-Scholze]**, $S \otimes_R^{\mathbf{L}} S \cong_q S \otimes_R S \cong S$, since ${}^{R}\!/_{\mathfrak{a}} \otimes_R {}^{R}\!/_{\mathfrak{a}} \cong {}^{R}\!/_{(\mathfrak{a} + \mathfrak{a})} \cong {}^{R}\!/_{\mathfrak{a}}$.

We observe that for any S-module P,
$$P \cong_q P \otimes_S^{\mathbf{L}} S \cong_q P \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} S) \cong_q (P \otimes_S^{\mathbf{L}} S) \otimes_R^{\mathbf{L}} S \cong_q P \otimes_R^{\mathbf{L}} S.$$

Thus we have
$$M \otimes_S^{\mathbf{L}} N \cong_q (M \otimes_S^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} N \cong_q M \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} N \cong_q (M \otimes S^{\mathbf{L}} S) \otimes_R^{\mathbf{L}} (S \otimes_S^{\mathbf{L}} N) \cong_q M \otimes_R^{\mathbf{L}} N.$$

The corollary thus follows. $\square$

Remark 1.3.45. Given this definition, a natural problem arises. We want a way to build perfect rings, in order to have more examples.

Definition 1.3.46 (perfection)**.** Fix any ring R of characteristic p, and set
$$R_{perf} = \varinjlim_{F} R$$
the **colimit perfection** of R. Note we have a natural map $R \to R_{perf}$. In fact, R_{perf} is a perfect ring, and for any perfect ring S with map $R \to S$, there is a diagram
$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R_{perf} \arrow[d, dashed] \\ & S \end{tikzcd}$$

Example 1.3.47. If $R = \mathbf{F}_p[x]$, then $R_{perf} = \mathbf{F}_p\left[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, ...\right] = \mathbf{F}_p\left[x^{\frac{1}{p^\infty}}\right]$.

Lemma 1.3.48. *If $R \to S$ is faithfully flat and S has finite Tor-dimension, then R has finite Tor-dimension.*

Remark 1.3.49. If R is local and characteristic $p > 0$, and the Frobenius F is flat, then the natural map $R \to R_{perf}$ is faithfully flat. (See [Stacks 00HP].)

Proof of ***Theorem 1.1.24 [Kunz]****, Part 2.* Assume $(R, \mathfrak{m})$ has a flat Frobenius F. We show that R has finite Tor-dimension; by **Theorem 1.3.38 [Auslander-Buchsbaum-Serre]**, the result follows. By **Remark 1.3.49**, $R \to R_{perf}$ is faithfully flat. By **Lemma 1.3.48**, it is enough to show that R_{perf} has finite Tor-dimension. Apply **Theorem 1.3.7 [Cohen Structure Theorem]** to write

$$\widehat{R_{perf}} \cong k[\![x_1, ..., x_d]\!]_{perf}/\mathfrak{a}.$$

By **Corollary 1.3.44**, it is enough to check that $k[\![x_1, ..., x_d]\!]_{perf}$ has finite Tor-dimension. Now, note the transition maps computing $k[\![x_1, ..., x_d]\!]_{perf}$ are flat (by the forward direction of **Theorem 1.1.24 [Kunz]**, proven in part 1). Hence any flat resolution of $k[\![x_1, ..., x_d]\!]_{perf}$-modules are also flat resolutions as $k[\![x_1, ..., x_d]\!]$-modules. That is, since $k[\![x_1, ..., x_d]\!]$ has finite Tor-dimension, $k[\![x_1, ..., x_d]\!]$ must have finite Tor-dimension, as desired. ☐

1.4 F-split Rings

By **Theorem 1.1.24 [Kunz]**, R is regular if and only if F_*R is free, so we observe singularties by observing "distance from free-ness." Let rings be of characteristic p and F-finite (i.e., F_*R is a finitely generated R-module).

Definition 1.4.1 (F-split)**.** A ring R is ***F*-split** provided the Frobenius map $R \to F_*R$ splits in R-**mod**. That is, there is an R-module map $F_*R \xrightarrow{\varphi} R$ so that the composition

$$\begin{array}{ccccc} R & \longrightarrow & F_*R & \xrightarrow{\varphi} & R \\ 1 & \longmapsto & F_*1 & \longmapsto & 1 \end{array}$$

is the identity.

Example 1.4.2. Let $R = \mathbf{F}_2[x]$. Then $F_*R \cong R^{\frac{1}{2}} \cong \mathbf{F}_2\left[x^{\frac{1}{2}}\right]$, a free R-module with basis $\left\{1, x^{\frac{1}{2}}\right\}$. Hence $\mathbf{F}_2\left[x^{\frac{1}{2}}\right] \cong R \cdot 1 \oplus R \cdot x^{\frac{1}{2}}$. The projection $\rho : R^{\frac{1}{2}} \cong R \cdot 1 \oplus R \cdot x^{\frac{1}{2}} \to R \cdot 1 \cong R$ splits the Frobenius.

Example 1.4.3. If R is regular, then F_*R is free by **Theorem 1.1.24 [Kunz]**, and $F_*R \cong \bigoplus_{i=1}^{p^d} R$ as R-modules, where $d = \dim R$. Then $R \to F_*R$ with $1 \mapsto F_*1$ has a splitting which is projection onto the F_*1-factor.

Remark 1.4.4. A splitting $\varphi \in \mathrm{Hom}_R(F_*R, R)$ sent through the natural map

$$\mathrm{Hom}_R(F_*R, R) \xrightarrow{ev_{F_*1}} R$$

maps $\varphi \mapsto \varphi(F_*1) = 1$. That is, if R is F-split, then ev_{F_*1} is surjective.

Lemma 1.4.5. *R is F-split if and only if ev_{F_*1} is surjective.*

Proof. By **Remark 1.4.4**, we only need to show that if ev_{F_*1} is surjective, then R is F-split. But indeed, suppose $\psi \in \mathrm{Hom}_R(F_*R, R)$ has $\psi(F_*1) = 1$. This forces ψ to be a splitting. ☐

Remark 1.4.6. For any map of R-modules $M \xrightarrow{\varphi} N$, φ is surjective if and only if $M_\mathfrak{m} \xrightarrow{\varphi_\mathfrak{m}} N_\mathfrak{m}$ is surjective for all $\mathfrak{m} \subseteq R$ maximal ideals. That is, using this fact and **Lemma 1.4.5**, the following are equivalent:
1. R is F-split,
2. ev_{F_*1} is surjective,
3. $(ev_{F_*1})_\mathfrak{m}$ is surjective for all maximal ideals $\mathfrak{m}$, and

4. $R_{\mathfrak{m}}$ is F-split for all maximal ideals $\mathfrak{m}$.

Remark 1.4.7. By **Theorem 1.1.24 [Kunz]**, every iterate of the Frobenius splits in a regular ring; i.e., for all $e \geq 1$ and R regular, $F_*^e R \cong \bigoplus_{i=1}^{p^{ed}} R$ has a projection $\varphi : F_*^e R \to R$ so that $R \to F_*^e R \xrightarrow{\varphi} R$ splits. This can actually be made stronger; in generality:

Lemma 1.4.8. *R is F-split if and only if $R \to F_*^e R$ splits for some (equivalently, for all) $e \geq 1$.*

Proof. If $R \to F_* R$ splits with $\varphi : F_* R \to R$, we can "iterate" φ by identifying $F_* R$ with R. That is, apply F_*- to $F_* R \xrightarrow{\varphi} R$. We get

$$F_*^2 R \xrightarrow{F_*\varphi} F_* R \xrightarrow{\varphi} R.$$

Thus, R is F-split implies $R \to F_*^e R$ splits for all e.

It now suffices to show that for a fixed $e > 0$, if $F_*^e R \xrightarrow{\varphi} R$ is a splitting, then R is F-split. Observe that the following map factors:

$$\begin{array}{ccccccc} R & \longrightarrow & F_* R & \longrightarrow & F_*^e R & \xrightarrow{\ \varphi\ } & R \\ 1 & \longmapsto & F_* 1 & \longmapsto & F_*^e 1 & \longmapsto & 1 \end{array}$$

Then $\psi : F_* R \to F_*^e R \xrightarrow{\varphi} R$ is a splitting for $R \to F_* R$, and hence R is F-split. ☐

Lemma 1.4.9. *Fix $R \subseteq S$ an extension of rings.*
1. *$R \to S$ splits if and only if there exists a surjective map $S \to R$.*
2. *If $R \to S$ splits and S is F-split, then R is F-split.*

Proof.
1. Certainly if $R \to S$ splits, then there is a surjective map $S \to R$, the splitting. Conversely, suppose $\psi : S \to R$ is surjective, and let $a \in S$ so that $\psi(a) = 1$. We construct a new map $\varphi : S \to R$ where $f \mapsto \psi(af)$. Thus $\varphi(-) = \psi(a-)$ is an R-module map, and therefore a splitting of $R \to S$. Observe

$$\begin{array}{ccccc} R & \longrightarrow & S & \xrightarrow{\qquad \varphi \qquad} & R \\ 1 & \longmapsto & 1 & \longmapsto & \varphi(1) = \psi(a) = 1 \end{array}$$

2. If S is F-split; i.e., there is a map $F_* S \xrightarrow{\varphi} S$, and if $R \subseteq S$ is split, i.e., there is a map $S \xrightarrow{\psi} R$, then the map

$$R \to F_* R \to F_* S \xrightarrow{\varphi} S \xrightarrow{\psi} R$$

is a splitting, so R is F-split.

☐

Example 1.4.10. Any direct summand of a regular ring is F-split.

Example 1.4.11. A Veronese is F-split: $k[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is a direct summand of $k[x, y]$.

Example 1.4.12. Fix a group G, and let G act on $S = k[x_1, ..., x_n]$ by homogeneous action. The invariants

$$S^G = \{f \in S \mid g \cdot f = f \text{ for all } g \in G\}$$

form a subring, and when $|G| \not\equiv 0 \bmod p$, then $S^G \subseteq S$ splits. The splitting is $\rho : S \to S^G$ defined by

$$\rho(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f$$

and is called the Reynolds operator.

Example 1.4.13. Let $p = 2$ and let $G = {}^{\mathbf{Z}}\!/_{2\mathbf{Z}} = \{e, g\}$. Let $S = k[x_1, y_1, x_2, y_2]$. The action is defined by

$$\begin{aligned} g \cdot x_1 &= x_1 + y_1 & \qquad g \cdot x_2 &= x_2 + y_2 \\ g \cdot y_1 &= y_1 & \qquad g \cdot y_2 &= y_2 \end{aligned}$$

In this case, $S^G = k[{x_1}^2 - y_1 x_1, y_1, {x_2}^2 - y_2 x_2, y_2, x_1 y_2 - x_2 y_1]$. We can check later (using **Corollary 1.4.24** [**Fedder's Criterion**]) that in this case, S^G is **not** F-split.

Example 1.4.14. Determinantal rings are F-split; i.e.,

$$R = {}^{k\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}\!\Big/_{I_2} = {}^{k[x,y,z,u,v,w]}\!\big/_{(xv - yu, xw - zu, yw - zv)},$$

where I_2 denotes the 2×2-minors of $\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$.

Remark 1.4.15. A natural question to ask is the following: when is a ring $R = {}^{k[x_1, ..., x_d]}\!/_{\mathfrak{a}}$ an F-split ring?

To answer this, set $S = k[x_1, ..., x_d]$, and we ask when a map $F_*S \to S$ descends to a map $F_*R \to R$. If this happens, then

$$\begin{tikzcd} F_*S \arrow[r, "\varphi"] \arrow[d, two heads] & S \arrow[d, two heads] \\ F_*R \arrow[r, "\psi"'] & R \end{tikzcd}$$

commutes. That is, we are asking for any given $\varphi \in \mathrm{Hom}_S(F_*S, S)$, when does $\varphi(F_*\mathfrak{a}) \subseteq \mathfrak{a}$? This would yield a well-defined ψ.

Definition 1.4.16 (φ-compatible)**.** Call an ideal $\mathfrak{a}$ a **φ-compatible** ideal if $\varphi(F_*^e\mathfrak{a}) \subseteq \mathfrak{a}$.

Remark 1.4.17. To get a better understanding, we consider $\mathrm{Hom}_S(F_*S, S)$ as a (F_*S)-module via the map $(F_*a) \cdot \varphi : F_*S \to S$, $F_*f \mapsto \varphi(F_*af)$.

Theorem 1.4.18. *If $S = k[x_1, ..., x_d]$ or a localization or a completion of $k[x_1, ..., x_d]$, then as an (F_*^eS)-module,*

$$\mathrm{Hom}_S(F_*^eS, S) \cong F_*^eS$$

with generator Φ_S^e given by

$$\Phi_S^e(F_*{x_1}^{\lambda_1} \cdots {x_d}^{\lambda_d}) = \begin{cases} 1 & \text{if } \lambda_i = p^e - 1 \text{ for all } i; \\ 0 & \text{otherwise.} \end{cases}$$

That is, Φ_S^e is projection onto the $({x_1}^{p^e-1}, ..., {x_d}^{p^e-1})$-factor.

Example 1.4.19. If $p = 2$ and $S = \mathbf{F}_2[x, y]$, then $F_*S \cong S^{\frac{1}{2}} \cong \bigoplus_{i=1}^4 S$ has basis $\left\{1, x^{\frac{1}{2}}, y^{\frac{1}{2}}, (xy)^{\frac{1}{2}}\right\}$, so

$$S^{\frac{1}{2}} \cong S \cdot 1 \oplus S \cdot x^{\frac{1}{2}} \oplus S \cdot y^{\frac{1}{2}} \oplus S \cdot (xy)^{\frac{1}{2}},$$

and Φ_S is projection onto $S \cdot (xy)^{\frac{1}{2}} \cong S$. We see that $\rho_x : S^{\frac{1}{2}} \to S$, defined by projection onto $S \cdot x^{\frac{1}{2}}$, is $\rho_x = y^{\frac{1}{2}} \cdot \Phi_S$. Similarly one can express all other maps $F_*S \to S$ in terms of the generator Φ_S.

Proof sketch of ***Theorem 1.4.18****.* First, write $F_*^eS \cong \bigoplus_{i=1}^{p^{ed}} S$ where $d = \dim S$, by **Theorem 1.1.24** [**Kunz**]. Now

$$\mathrm{Hom}_S(F_*^eS, S) \cong \mathrm{Hom}_S\left(\bigoplus_{i=1}^{p^{ed}} S, S\right) \cong \bigoplus_{i=1}^{p^{ed}} \mathrm{Hom}_S(S, S) \cong \bigoplus_{i=1}^{p^{ed}} S \cong F_*^eS.$$

It suffices to prove that each projection $F_*^eS \to S$ has the form $F_*^eu \cdot \Phi_S^e$. $\square$

Remark 1.4.20. If $\varphi \in \operatorname{Hom}_S(F^e_* S, S)$ and $g \in \mathfrak{a}^{[p^e]}$ (recall **Definition 1.1.19 [Frobenius power]**), then $\mathfrak{a} \cdot F^e_* S = F^e_* \mathfrak{a}^{[p^e]}$ and $F^e_* g = h \cdot F^e_* 1$ for $h \in \mathfrak{a}$. Thus,

$$\varphi(F^e_* g) = \varphi(h \cdot F^e_* 1) = h \cdot \varphi(F^e_* 1) \in \mathfrak{a}.$$

This means that any $u \in \left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) = \left\{f \in S \mid f\mathfrak{a} \subseteq \mathfrak{a}^{[p^e]}\right\}$ will force $\varphi = F^e_* u \cdot \Phi^e_S$ to satisfy $\varphi(F^e_* \mathfrak{a}) \subseteq \mathfrak{a}$; i.e., will force $\mathfrak{a}$ to be φ-compatible.

Let $x \in \mathfrak{a}$. We have $\varphi(F^e_* x) = \Phi^e_S(F^e_* u \cdot x) = \Phi^e_S(y \cdot F^e_* 1) \in \mathfrak{a}$, with $y \in \mathfrak{a}$. This gives a natural homomorphism as $F^e_* S$-modules:

$$F^e_* \left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \cdot \operatorname{Hom}_S(F^e_* S, S) \to \operatorname{Hom}_R(F^e_* R, R).$$

Lemma 1.4.21. *Fix $R = S/\mathfrak{a}$. Let $\mathfrak{b} \subseteq S$ be any ideal. If $\varphi \in F^e_* \left(\mathfrak{a}^{[p^e]} : \mathfrak{b}\right) \cdot \operatorname{Hom}_S(F^e_* S, S)$, then φ satisfies $\varphi(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}$. Moreover, if $\varphi(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}$ for all $\varphi \in \operatorname{Hom}_S(F^e_* S, S)$, then $\mathfrak{b} \subseteq \mathfrak{a}^{[p^e]}$. In particular,*

$$F^e_* \left(\mathfrak{a}^{[p^e]} : \mathfrak{b}\right) \cdot \operatorname{Hom}_S(F^e_* S, S) = \{\varphi \in \operatorname{Hom}_S(F^e_* S, S) \mid \varphi(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}\}.$$

Proof. Let $\varphi \in F^e_* \left(\mathfrak{a}^{[p^e]} : \mathfrak{b}\right) \cdot \operatorname{Hom}_S(F^e_* S, S)$. By a direct generalization of **Remark 1.4.20** above, we have $\varphi(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}$.

Next, assume that $\varphi(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}$ for all $\varphi \in \operatorname{Hom}_S(F^e_* S, S)$. Recall that by **Theorem 1.1.24 [Kunz]**, $F^e_* S \cong \bigoplus_{i=1}^{p^{ed}} S$, so for any projection ρ, $\rho(F^e_* \mathfrak{b}) \subseteq \mathfrak{a}$ by hypothesis. Thus

$$F^e_* \mathfrak{b} \subseteq \mathfrak{a} F^e_* S = F^e_* \mathfrak{a}^{[p^e]} \cong \bigoplus_{i=1}^{p^{ed}} \mathfrak{a},$$

which forces $\mathfrak{b} \subseteq \mathfrak{a}^{[p^e]}$. Indeed, let $x \in \mathfrak{b}$. Apply all projections $\rho(F^e_* x) \in \mathfrak{a}$. If we write $F^e_* x = (x_{\lambda_1,\dots,\lambda_d})_{\lambda_1,\dots,\lambda_d} \in \bigoplus_{i=1}^{p^{ed}} S$, then in any slot, $x_{\lambda_1,\dots,\lambda_d} \in \mathfrak{a}$. Hence, $F_* \mathfrak{b} \subseteq \bigoplus_{i=1}^{p^{ed}} \mathfrak{a} \cong \mathfrak{a} \cdot F^e_* S \cong F^e_* \mathfrak{a}^{[p^e]}$. Therefore, $\mathfrak{b} \subseteq \mathfrak{a}^{[p^e]}$, as claimed. ☐

Theorem 1.4.22 (Fedder's Lemma)**.** *Fix $R = S/\mathfrak{a}$ with S a polynomial ring. The map*

$$F^e_* \left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \cdot \operatorname{Hom}_S(F^e_* S, S) \to \operatorname{Hom}_R(F^e_* R, R)$$

is surjective with kernel $F^e_* \left(\mathfrak{a}^{[p^e]}\right) \cdot \operatorname{Hom}_S(F^e_* S, S)$. That is, as $F^e_* S$-modules,

$$\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* \left(\left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \Big/ \mathfrak{a}^{[p^e]} \right).$$

Proof. Surjectivity: BLANK.

To confirm the kernel, apply **Lemma 1.4.21** with $\mathfrak{b} = S$; the result follows. ☐

Example 1.4.23. The condition that S is regular is **necessary**. If we let $S = k[x, y, z]$, $T = S/(x^2 - yz)$, and $R = T/(x, y) \cong k[z]$, then the map $\varphi : F_* R \to R$ sending $F_* z^{p-1} \mapsto 1$ does **not** lift to T. First, lift φ to S. We need an element $u \in ((x^p, y^p) : (xy)) = (xy)^{p-1} + (x^p, y^p)$, and hence we get $F_* u \cdot \Phi_S$. But to lift this map to T, we would need $v \in \left((x^2 - yz)^p : (x, y)\right)$.

Corollary 1.4.24 (Fedder's Criterion)**.** *Let $(S, \mathfrak{m}) = k[x_1, \dots, x_d]_{\mathfrak{m}}$. The following are equivalent:*

1. *$R = S/\mathfrak{a}$ is F-split,*
2. *$\left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \not\subseteq \mathfrak{m}^{[p^e]}$ for some e, and*
3. *$\left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \not\subseteq \mathfrak{m}^{[p^e]}$ for all e.*

Proof. The equivalence between 2 and 3 is **Lemma 1.4.8**.

If $R = {}^{S}\!/_{\mathfrak{a}}$ is F-split, then $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ with $\varphi(F^e_*R) \not\subseteq \mathfrak{m}$. Let $f \in R$ such that $\varphi(F^e_*f) \notin \mathfrak{m}$. By **Theorem 1.4.22 [Fedder's Lemma]**, $\varphi = F^e_*u \cdot \Phi^e_S$ for $u \in \left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right)$. If $\left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \subseteq \mathfrak{m}^{[p^e]}$, then $\varphi(F^e_*f) = \Phi^e_S(F^e_*uf) \in \mathfrak{m}$, a contradiction. Hence 1 implies 2.

Conversely, if $g \in \left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \setminus \mathfrak{m}^{[p^e]}$, then $F^e_*g \cdot \Phi^e_S$ descends to a map $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ and $\varphi(F^e_*R) \not\subseteq \mathfrak{m}$. ☐

Corollary 1.4.25. *If $\mathfrak{a}$ is principle; i.e., $\mathfrak{a} = (f)$, then ${}^{S}\!/_{(f)}$ is F-split if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$.*

Example 1.4.26. Let $S = k[x, y]$ and $R = {}^{S}\!/_{(xy)}$. Then R is F-split, as $(xy)^{p-1} \notin (x^p, y^p)$.

Remark 1.4.27. We finally have an answer to the question posed in **Remark 1.4.15**: for which maps $\varphi \in \mathrm{Hom}_S(F_*S, S)$ is $\mathfrak{a}$ φ-compatible? We can ask another question: fix $\varphi \in \mathrm{Hom}_S(F_*S, S)$. Which $\mathfrak{a}$ are φ-compatible?

Theorem 1.4.28 (Schwede)**.** *If S is any F-split ring and $\varphi \in \mathrm{Hom}_S(F_*S, S)$, then there are only finitely many $\mathfrak{a}$ that are φ-compatible.*

1.4.1 Symbolic Powers

Definition 1.4.29 (primary)**.** If R is any noetherian ring, then an ideal $\mathfrak{q} \subseteq R$ is called **primary** provided $x \cdot y \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y \in \sqrt{\mathfrak{q}}$.

Lemma 1.4.30. *If $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime.*

Proof. Let $ab \in \sqrt{\mathfrak{q}}$. Thus $(ab)^n = a^n b^n \in \mathfrak{q}$ for some n. Thus $a^n \in \mathfrak{q}$ or $b^n \in \sqrt{\mathfrak{q}}$. Thus $a \in \sqrt{\mathfrak{q}}$ or $b \in \sqrt{\sqrt{\mathfrak{q}}} = \sqrt{\mathfrak{q}}$. Thus $\sqrt{\mathfrak{q}}$ is prime. ☐

Definition 1.4.31 ($\mathfrak{p}$-primary)**.** If $\mathfrak{q}$ is primary and $\mathfrak{p} = \sqrt{\mathfrak{q}}$, then $\mathfrak{q}$ is called a **$\mathfrak{p}$-primary ideal**.

Example 1.4.32. In $\mathbf{Z}$, (p^n) is a (p)-primary ideal.

Example 1.4.33. If $(S, \mathfrak{m})$ is a local ring, then $\mathfrak{m}^n$ are $\mathfrak{m}$-primary.

☡ **Warning! 1.4.34.** Not all prime powers are primary. If $R = {}^{k[x,y,z]}\!/_{(xz - y^2)}$ and $\mathfrak{p} = (x, y)$, then $\mathfrak{q} = \mathfrak{p}^2 = (x^2, xy, y^2)$ is not primary. Observe that $xz = y^2 \in \mathfrak{q}$, but $x \notin \mathfrak{q}$ and $z^n \notin \sqrt{\mathfrak{q}}$ for any n.

Remark 1.4.35. Recall that in $\mathbf{Z}$, a prime factorization $n = {p_1}^{e_1} \cdots {p_j}^{e_j}$ forces an equality of ideals

$$(n) = ({p_1}^{e_1}) \cap \cdots \cap ({p_j}^{e_j}).$$

Each $({p_i}^{e_i})$ is (p_i)-primary.

Definition 1.4.36 (primary decomposition)**.** For a noetherian ring R, a **primary decomposition** of an ideal $\mathfrak{a}$ is $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ with each $\mathfrak{q}_i$ primary.

Definition 1.4.37 (irredundant)**.** We call a primary decomposition of an ideal $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ **irredundant** if no $\mathfrak{q}_i$ can be removed.

Definition 1.4.38 (associated primes)**.** Given a primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, we call $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ the **associated primes** of $\mathfrak{a}$. We write $\mathrm{Ass}\left({}^{R}\!/_{\mathfrak{a}}\right) = \{\mathfrak{p}_i \mid \mathfrak{p}_i = \sqrt{\mathfrak{q}_i} \text{ is an associated prime}\}$. (Equivalently when R is commutative, $\mathfrak{p}$ is an associated prime of an R-module M if ${}^{R}\!/_{\mathfrak{p}}$ is isomorphic to a submodule of M; we write $\mathfrak{p} \in \mathrm{Ass}(M)$.)

Definition 1.4.39 (minimal primes)**.** Given a primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ with associated primes $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, we call the minimal $\mathfrak{p}_i$s with respect to inclusion the **minimal primes**.

Definition 1.4.40 (embedded primes)**.** Given a primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ with associated primes $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, if $\mathfrak{p}_i$ is not a minimal prime, then we call it an **embedded prime**.

Example 1.4.41. If $R = k[x,y]_{(x,y)}$, then $(x^2, xy) = (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, xy, y^3) = \cdots$. The minimal prime is (x), while the embedded prime is (x, y).

Theorem 1.4.42 (Noether)**.** *In a noetherian ring, every ideal $\mathfrak{a}$ has an irredundant primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, and the set of minimal primes is unique.*

Definition 1.4.43 (symbolic power)**.** Let $\mathfrak{a}$ be an ideal without embedded primes. Set

$$\mathfrak{a}^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}\left(R/\mathfrak{a}\right)} \left(\mathfrak{a}^n R_{\mathfrak{p}} \cap R\right),$$

and call $\mathfrak{a}^{(n)}$ the n^{th} **symbolic power** of $\mathfrak{a}$.

Remark 1.4.44. We have $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)}$, but equality easily fails.

Example 1.4.45. If $\mathfrak{p} = (x^2y - z^2, xz - y^2, yz - x^3) \subseteq k[x,y,z]$, then $\mathfrak{p}^{(n)} \neq \mathfrak{p}^n$ for $n \geq 2$. In fact, $\mathfrak{p}^{(2)} \not\subseteq \mathfrak{p}^2$, but $\mathfrak{p}^{(3)} \subseteq \mathfrak{p}^2$.

Remark 1.4.46. We now have a natural question: for any fixed ideal $\mathfrak{a}$, when does $\mathfrak{a}^{(k)} \subseteq \mathfrak{a}^n$ hold?

Remark 1.4.47. Due to a result by Schenzel, symbolic powers are cofinal with ordinary powers. In fact, for each n, there is an integer c such that $\mathfrak{a}^{(cn)} \subseteq \mathfrak{a}^n$, and c can be chosen independent of $\mathfrak{a}$! (Though, c still depends on R.) That is, the discrepancy is "linear." This result is due to Swanson.

Definition 1.4.48 (big height)**.** For a radical ideal $\mathfrak{a}$, define the **big height** of $\mathfrak{a}$ to be

$$\operatorname{bight}(\mathfrak{a}) = \max_{\mathfrak{p} \in \operatorname{Ass}\left(R/\mathfrak{a}\right)} \operatorname{ht} \mathfrak{p}.$$

Example 1.4.49. If $\mathfrak{a} = (xy, xz) = (x) \cap (y, z)$, then $\mathfrak{a}$ has associated primes (x) and (y, z). We have $\operatorname{ht} \mathfrak{a} = 1$, while $\operatorname{bight} \mathfrak{a} = 2$.

Theorem 1.4.50 (Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede)**.** *Let R be a regular ring and let $\mathfrak{a}$ be a radical ideal with $\operatorname{bight} \mathfrak{a} = h$. If $n \geq 1$, then $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

Remark 1.4.51. Harborne asked: for a homogeneous radical ideal $\mathfrak{a}$ in a polynomial ring with $\operatorname{bight} \mathfrak{a} = h$, is it the case that $\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n$? Unfortunately this fails in general, due to a result by Harborne-Seceleanu.

Remark 1.4.52. Our goal is to show that $\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n$ holds if the ring $S/\mathfrak{a}$ is F-split.

Theorem 1.4.53 (Hochster-Huneke)**.** *For a radical ideal $\mathfrak{a}$ in a polynomial ring S with $\operatorname{bight} \mathfrak{a} = h$ and $q = p^e \geq p$, we have $\mathfrak{a}^{(hq)} \subseteq \mathfrak{a}^{[q]}$.*

Proof. As S is regular, by **Theorem 1.1.24 [Kunz]** the Frobenius on S is flat. This forces $\operatorname{Ass}\left(S/\mathfrak{a}\right)$ and $\operatorname{Ass}\left(S/\mathfrak{a}^{[q]}\right)$ to be the same. Fix an associated prime $\mathfrak{p}$, and note that $\mathfrak{p}$ is generated by at most h elements in $S_{\mathfrak{p}}$, by definition of $\operatorname{bight} \mathfrak{a}$. Set $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a} S_{\mathfrak{p}}$. It suffices to check that ${\mathfrak{a}_{\mathfrak{p}}}^{hq} \subseteq \mathfrak{p}^{[q]}$.

Indeed, write $\mathfrak{a}_{\mathfrak{p}} = (x_1, ..., x_h)$. We need to show that $(x_1, ..., x_h)^{hq} \subseteq ({x_1}^q, ..., {x_h}^q)$; the Pigeonhole Principle does the job. $\square$

Theorem 1.4.54 (Hochster-Huneke)**.** *Let S be a regular ring of characteristic p. Let $\mathfrak{a}$ be a radical ideal with* bight $\mathfrak{a} = h$*. For all $n \geq 0$ and $t > 0$, $\mathfrak{a}^{(hn+tn)} \subseteq \left(\mathfrak{a}^{(t+1)}\right)^n$.*

Theorem 1.4.55 (Grifo-Huneke)**.** *If S is a regular ring, $\mathfrak{a}$ is a radical ideal with* bight $\mathfrak{a} = h$*, and ${S}/{\mathfrak{a}}$ is F-split, then for $n \geq 1$, $\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n$.*

Proof. Without loss of generality, we can assume S is local. It suffices to prove the following:

Claim. *For $q = p^e \gg 0$, $\left(\mathfrak{a}^{[q]} : \mathfrak{a}\right) \subseteq \left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right)^{[q]}$.*

Proof. As the Frobenius is flat by **Theorem 1.1.24 [Kunz]**, we have

$$\left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right)^{[q]} = \left((\mathfrak{a}^n)^{[q]} : \left(\mathfrak{a}^{(hn-h+1)}\right)^{[q]}\right).$$

Set $f \in \left(\mathfrak{a}^{[q]} : \mathfrak{a}\right)$. We wish to show that $f \cdot \left(\mathfrak{a}^{(hn-h+1)}\right)^{[q]} \subseteq (\mathfrak{a}^n)^{[q]}$. Note that

$$f \cdot \mathfrak{a}^{(hn-h+1)} \subseteq f \cdot \mathfrak{a} \subseteq \mathfrak{a}^{[q]}.$$

So,

$$\begin{aligned} f \cdot \left(\mathfrak{a}^{(hn-h+1)}\right)^{[q]} &\subseteq f \cdot \left(\mathfrak{a}^{(hn-h+1)}\right)^{q} \\ &= \left(f \cdot \mathfrak{a}^{(hn-h+1)}\right) \cdot \left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \\ &\subseteq \mathfrak{a}^{[q]} \cdot \left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1}. \end{aligned}$$

Thus, to prove the claim, it suffices to show $\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \subseteq \left(\mathfrak{a}^{[q]}\right)^{n-1}$, since $\left(\mathfrak{a}^{[q]}\right)^n = (\mathfrak{a}^n)^{[q]}$. To that end, pick q large so that $(hn-h+1)(q-1) \geq h(n-1) + (hq-1)(n-1)$. Thus

$$\left(\mathfrak{a}^{(hn-1+1)}\right)^{q-1} \subseteq \left(\mathfrak{a}^{(hn-h+1)}\right)^{(q-1)} \subseteq \mathfrak{a}^{((hn-h+1)(q-1))} \subseteq \mathfrak{a}^{(h(n-1)+(hq-1)(n-1))}.$$

Therefore,

$$\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \subseteq \mathfrak{a}^{(h(n-1)+(hq-1)(n-1))} \subseteq \left(\mathfrak{a}^{(hq)}\right)^{n-1} \subseteq \left(\mathfrak{a}^{[q]}\right)^{n-1},$$

as desired. □

See that the claim yields the theorem, as $\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n$ if and only if $\left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right) = S$. We then see that, assuming to the contrary that $\left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right) \subseteq \mathfrak{m}$, the claim implies that

$$\left(\mathfrak{a}^{[q]} : \mathfrak{a}\right) \subseteq \left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]},$$

contradicting that ${S}/{\mathfrak{a}}$ is F-split by **Corollary 1.4.24 [Fedder's Criterion]**. □

1.4.2 Geometric Perspective

Remark 1.4.56. Fix a ring R. Recall (**Definition 1.1.2 [spectrum]**, **Definition 1.1.3 [scheme]**) that we get an affine scheme $X = \operatorname{Spec} R$ which is a locally ringed space. X has a structure sheaf $\mathcal{O}_X$, and we have the global sections functor $\Gamma(X, -)$ for which $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) \cong R$. The Frobenius $F : X \to X$ coming functorially from $F : R \to R$ is an identity on points of X, but it takes $\mathcal{O}_X$ to $F_*\mathcal{O}_X$.

Definition 1.4.57 (direct image functor)**.** Given a continuous map of underlying topological spaces $f : X \to Y$, we define the **direct image functor** f_* from sheaves on X to sheaves on Y to send a sheaf $\mathcal{F}$ on X to $f_*\mathcal{F}$, the (pre)sheaf on Y for which, given $V \subseteq Y$, $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$.

Remark 1.4.58. We restate **Theorem 1.1.24 [Kunz]** in geometric language:

Theorem 1.4.59 (Kunz)**.** *If $X = \operatorname{Spec} R$, then the following are equivalent:*

1. *R is regular,*
2. *$F : X \to X$ is flat, and*
3. *$F_*\mathcal{O}_X$ is locally free; i.e., for every point $x \in X$ there exists an open neighborhood U of x such that*

$$F_*\mathcal{O}_X|_U \cong \bigoplus_{i \in I} \mathcal{O}_X|_U$$

as $\mathcal{O}_X|_U$-modules for I some indexing set.

Definition 1.4.60 (globally F-split)**.** Call a scheme X **globally F-split** if $\mathcal{O}_X \to F_*\mathcal{O}_X$ is split as a map of $\mathcal{O}_X$-modules.

Remark 1.4.61. It is evident that when $X = \operatorname{Spec} R$, the following are equivalent:

1. X is globally F-split,
2. R is F-split, and
3. $R_\mathfrak{m}$ is F-split for all $\mathfrak{m}$

(by **Remark 1.4.6**). However, in general, if X is a scheme, then each local ring $\mathcal{O}_{X,x_0}$ being F-split need not imply that X is globally F-split.

Theorem 1.4.62. *Let S be a regular ring. Let $\mathfrak{a} \subseteq S$ be an ideal, and let $X = \operatorname{Spec}\left({}^{S}\!/\!_{\mathfrak{a}}\right)$. Define an ideal*

$$\mathfrak{b}_e = \operatorname{im}\left(F_*^e\left(\mathfrak{a}^{[p^e]} : \mathfrak{a}\right) \cdot \operatorname{Hom}_S\left(F_*^e S, S\right) \xrightarrow{ev_{F_*1}} S\right).$$

The set theoretic locus $V(\mathfrak{b}_e) \subseteq V(\mathfrak{a}) \cong X \subseteq \operatorname{Spec} S$ is the set of points where X is not globally F-split. That is, the globally F-split locus of X is open.

Remark 1.4.63. The ideal $\sqrt{\mathfrak{b}_e}$ does not depend on e, but $\mathfrak{b}_e$ itself does. That is, the embedded primes/scheme structure of $\operatorname{Spec}\left({}^{S}\!/\!_{\mathfrak{b}_e}\right)$ do depend on e.

Remark 1.4.64. We also restate **Theorem 1.4.28 [Schwede]** in geometric language:

Theorem 1.4.65 (Schwede)**.** *If $X = \operatorname{Spec} R$ is globally F-split, then there are only finitely many compatibly F-split subschemes of X.*

Proof sketch by means of decoding the geometric statement into algebra. A subscheme of X that is of the form $Y = \operatorname{Spec}\left({}^{R}\!/\!_{\mathfrak{a}}\right)$ is called compatibly F-split if $\varphi : F_*R \to R$ is a splitting and $\mathfrak{a}$ is a φ-compatible ideal. Hence, the theorem is equivalent to **Theorem 1.4.28 [Schwede]**. ☐

1.5 Local Cohomology

Fix a ring R and an ideal $\mathfrak{a} \subseteq R$.

Definition 1.5.1 (artinian module)**.** An **artinian** module M satisfies the descending chain condition on submodules. That is, there is no infinite descending chain of submodules $M = N_0 \supsetneq N_1 \supsetneq \cdots$. In other words, given an infinite chain $N_0 \supseteq N_1 \supseteq \cdots$, there exists $n \in \mathbf{N}$ such that $N_n = N_k$ for all $k \geq n$.

Definition 1.5.2 ($\mathfrak{a}$-torsion)**.** The functor $\Gamma_{\mathfrak{a}} : R$-**mod** $\to R$-**mod** for which

$$M \mapsto \Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^t m = 0\} = \bigcup_{t \geq 0} \left(0 :_M \mathfrak{a}^t\right)$$

is the $\mathfrak{a}$**-torsion** functor, and $\Gamma_{\mathfrak{a}}(M)$ is the $\mathfrak{a}$-torsion submodule of M.

Remark 1.5.3. The functor $\Gamma_{\mathfrak{a}}$ is left, but not right, exact; i.e., given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, the sequence $0 \to \Gamma_{\mathfrak{a}}(A) \to \Gamma_{\mathfrak{a}}(B) \to \Gamma_{\mathfrak{a}}(C)$ is exact. Hence we may elevate $\Gamma_{\mathfrak{a}}$ to the derived functors $\mathbf{R}^i\Gamma_{\mathfrak{a}}(M) = h^i(\Gamma_{\mathfrak{a}}(E^\bullet))$ for $0 \to M \to E^\bullet$ any injective resolution. In the derived category $D(R)$, we have the total derived functor $\mathbf{R}\Gamma_{\mathfrak{a}} : D(R) \to D(R)$ for which $h^i(\mathbf{R}\Gamma_{\mathfrak{a}}([M])) \cong \mathbf{R}^i\Gamma_{\mathfrak{a}}(M)$.

Definition 1.5.4 (local cohomology)**.** We define the i^{th} **local cohomology** of an R-module M to be $H^i_{\mathfrak{a}}(M) = \mathbf{R}^i(\Gamma_{\mathfrak{a}}(M))$.

Remark 1.5.5. We have the following facts about local cohomology:

1. If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then there is an exact triangle
$$\mathbf{R}\Gamma_{\mathfrak{a}}(A) \to \mathbf{R}\Gamma_{\mathfrak{a}}(B) \to \mathbf{R}\Gamma_{\mathfrak{a}}(C) \xrightarrow{+1} \mathbf{R}\Gamma_{\mathfrak{a}}(A);$$
i.e., there is a long exact sequence
$$\cdots \to H^i_{\mathfrak{a}}(A) \to H^i_{\mathfrak{a}}(B) \to H^i_{\mathfrak{a}}(C) \to H^{i+1}_{\mathfrak{a}}(A) \to \cdots .$$
2. The natural Frobenius $F : R \to R$ induces additive maps $F : H^i_{\mathfrak{a}}(R) \to H^i_{\mathfrak{a}}(R)$.
3. For $i > \dim R$ or $i < 0$, $H^i_{\mathfrak{a}}(M) = 0$ for all M.
4. $\mathbf{R}\Gamma_{\mathfrak{a}} \cong \mathbf{R}\Gamma_{\mathfrak{b}}$ as functors if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
5. $\mathbf{R}\Gamma_{\mathfrak{a}}$ is additive; i.e., $\mathbf{R}\Gamma_{\mathfrak{a}}\left(\bigoplus_i M_i\right) \cong \bigoplus_i \mathbf{R}\Gamma_{\mathfrak{a}}(M_i)$.
6. One may identify $H^i_{\mathfrak{a}}(M) \cong \varinjlim_t \operatorname{Ext}^i\left(R/\mathfrak{a}^t, M\right)$, and for any cofinal system, the limit does not change; i.e., if R has characteristic $p > 0$, then $H^i_{\mathfrak{a}}(M) \cong \varinjlim_e \operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, M\right)$.
7. If $R \hookrightarrow S$ is any inclusion of rings and N is an S-module, then $\mathbf{R}\Gamma_{\mathfrak{a}}(N) \cong \mathbf{R}\Gamma_{\mathfrak{a}S}(N)$. If $R \hookrightarrow S$ is flat, then for an R-module M, $S \otimes_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \cong \mathbf{R}\Gamma_{\mathfrak{a}S}(S \otimes_R M)$. That is, for each i,
$$S \otimes_R H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}S}(S \otimes_R M).$$
8. For a directed system $\{M_j\}_{j \in \mathbf{N}}$ of R-modules, $H^i_{\mathfrak{a}}\left(\varinjlim_j M_j\right) \cong \varinjlim_j H^i_{\mathfrak{a}}(M_j)$.
9. If $(R, \mathfrak{m})$ is local and M is finitely generated, then each $H^i_{\mathfrak{m}}(M)$ is artinian.
10. $H^i_{\mathfrak{a}}(-)$ is an analog of cohomology of topological spaces with supports. For each R-module M, set $U = \operatorname{Spec} R \setminus V(\mathfrak{a})$, and set $\widetilde{M}$ to be the sheaf associated to M on $\operatorname{Spec} R$ (that is, for $D_f \subseteq X$ a standard open (recall **Definition 1.1.2 [spectrum]**), $\widetilde{M}(D_f) = M_f \cong M \otimes_R R_f$). There is an exact sequence
$$0 \to H^0_{\mathfrak{a}}(M) \to M \to H^0(U, \widetilde{M}) \to H^1_{\mathfrak{a}}(M) \to 0$$
and isomorphisms $H^i_{\mathfrak{a}}(M) \cong H^{i+1}(U, \widetilde{M})$. More generally, for a topological space X and a closed subspace $Y \subseteq X$, the functor $\Gamma_Y(X, -)$ takes a sheaf on X to global sections $s \in \Gamma(X, \mathcal{F})$ with stalks $s_{x_0} = 0$ for $x_0 \in X \setminus Y$. The functor $\Gamma_Y(X, -)$ is left exact, and $H^i_Y(X, -) = h^i(\mathbf{R}\Gamma_Y(X, -))$ is cohomologically supported in Y. Apply this to $X = \operatorname{Spec} R$, $Y = V(\mathfrak{a})$, and $\mathcal{F} = \widetilde{M}$; we recover $H^i_{\mathfrak{a}}(M)$.

Definition 1.5.6 (annihilator)**.** Let M be an R-module, and $S \subseteq M$ a subset. The **annihilator** of S is $\operatorname{Ann}_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$.

Remark 1.5.7. A fundamental result of Grothendieck ensures that if $(R,\mathfrak{m})$ is a local ring, M is a finitely generated R-module, and $i > \dim M = \dim \left(R/\operatorname{Ann} M \right)$, then $H^i_{\mathfrak{m}}(M) = 0$. Furthermore, $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$. In particular, $H^{\dim R}_{\mathfrak{m}}(R) \neq 0$.

Remark 1.5.8. There is a notion of Čech complexes for $H^i_{\mathfrak{a}}(M)$. Fix a generating set $(f_1, ..., f_s)$ for $\mathfrak{a}$; one has the complex

$$\check{\mathrm{C}}^\bullet(f_1, ..., f_s; M) : \qquad 0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to \bigoplus_i M_{f_1 \cdots \widehat{f_i} \cdots f_s} \to M_{f_1 \cdots f_s} \to 0.$$

One may prove that $h^i(\check{\mathrm{C}}^\bullet(f_1, ..., f_s; M)) \cong H^i_{\mathfrak{a}}(M)$. It is therefore immediate that $H^i_{\mathfrak{a}}(M) = 0$ for $i > s$.

Remark 1.5.9. If $H^i_{\mathfrak{a}}(M) \neq 0$, then $\sqrt{\mathfrak{a}} = (f_1, ..., f_s)$ with s minimal must have $i \leq s$. The Čech complex gives a very explicit representation of

$$H^s_{(f_1,...,f_s)}(R) \cong R_{f_1 \cdots f_s} \Big/ \operatorname{im}\left(\bigoplus_i R_{f_1 \cdots \widehat{f_i} \cdots f_s} \to R_{f_1 \cdots f_s} \right).$$

So an element $\eta \in H^s_{(f_1,...,f_s)}(R)$ is an equivalence class $\eta = \left[\frac{g}{{f_1}^a \cdots {f_s}^a} \right]$.

Lemma 1.5.10. *For a ring R, $\mathfrak{a} = (f_1, ..., f_s)$, a class $\eta = \left[\frac{g}{{f_1}^a \cdots {f_s}^a} \right] \in H^s_{\mathfrak{a}}(R)$ is zero if and only if there is a non-negative integer k such that $g(f_1 \cdots f_s)^k \in \left({f_1}^{a+k}, ..., {f_s}^{a+k} \right)$.*

Proof sketch. We only prove that the existence of such a k implies that $\eta = 0$. Write $g(f_1 \cdots f_s)^k = \sum r_i {f_i}^{a+k}$. Observe that

$$\begin{aligned}
\eta = \left[\frac{g}{{f_1}^a \cdots {f_s}^a} \right] &= \left[\frac{g({f_1}^k \cdots {f_s}^k)}{{f_1}^{a+k} \cdots {f_s}^{a+k}} \right] \\
&= \left[\frac{\sum r_i {f_i}^{a+k}}{{f_1}^{a+k} \cdots {f_s}^{a+k}} \right] \\
&= \sum r_i \left[\frac{{f_i}^{a+k}}{{f_1}^{a+k} \cdots {f_s}^{a+k}} \right] \\
&= \sum r_i \left[\frac{1}{{f_1}^{a+k} \cdots \widehat{{f_i}^{a+k}} \cdots {f_s}^{a+k}} \right] \in \operatorname{im}\left(\bigoplus_i R_{f_1 \cdots \widehat{f_i} \cdots f_s} \to R_{f_1 \cdots f_s} \right),
\end{aligned}$$

so $\eta = 0$, as claimed. □

Remark 1.5.11. Additionally, for $\sqrt{\mathfrak{a}} = (f_1, ..., f_s)$, one may express $H^s_{\mathfrak{a}}(R) = \varinjlim_m R/({f_1}^m, ..., {f_s}^m)$, where the transition maps are multiplication by $f_1 \cdots f_s$.

Example 1.5.12. Let $R = k[x]$ and $\mathfrak{m} = (x)$. We have the Čech complex

$$\check{\mathrm{C}}(x; R) : \qquad 0 \to R \to R_x = R\left[x^{-1}\right] \to 0.$$

So $H^0_{\mathfrak{m}}(R) = 0$, and $H^1_{\mathfrak{m}}(R) = R_x/R \cong k[x, x^{-1}]/k[x] \cong x^{-1}k[x^{-1}]$ is a k-vector space with basis $\{x^{-1}, x^{-2}, x^{-3}, ...\}$ and R-action

$$x^a \left(\frac{1}{x^n} \right) = \begin{cases} \frac{1}{x^{n-a}} & \text{if } a < n; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $H^1_{\mathfrak{m}}(R)$ is **not** a finitely generated R-module.

Remark 1.5.13. More generally, if $R = k[x_1, ..., x_d]$ and $\mathfrak{m} = (x_1, ..., x_d)$, then $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$ and $H^{\dim R}_{\mathfrak{m}}(R)$ is the k-span of $\left\{ \frac{1}{{x_1}^{a_1} \cdots {x_d}^{a_d}} \mid a_i > 0 \right\}$.

Example 1.5.14. When $d = 2$, we have

$$\begin{tikzcd}
 & & & \vdots & & \\
 & x^2 & & xy & & y^2 \\
k[x,y]: & & x \arrow[ul, "\cdot x"] \arrow[ur, "\cdot y"] & & y \arrow[ul, "\cdot x"'] \arrow[ur, "\cdot y"'] & \\
 & & & 1 \arrow[ul, "\cdot x"] \arrow[ur, "\cdot y"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & \frac{1}{xy} & & \\
H^2_{(x,y)}(k[x,y]): & & \frac{1}{x^2y} \arrow[ur, "\cdot x"] & & \frac{1}{xy^2} \arrow[ul, "\cdot y"'] & \\
 & \frac{1}{x^3y} \arrow[ur, "\cdot x"] & & \frac{1}{x^2y^2} \arrow[ul, "\cdot y"] \arrow[ur, "\cdot x"'] & & \frac{1}{xy^3} \arrow[ul, "\cdot y"'] \\
 & & & \vdots & &
\end{tikzcd}$$

Definition 1.5.15 (depth)**.** For a local ring $(R, \mathfrak{m})$ and R-module M, we define the **depth** of M to be

$$\operatorname{depth} M = \min_n \left\{ H^n_{\mathfrak{m}}(M) \neq 0 \right\}.$$

Remark 1.5.16. By definition, $\mathbf{R}\Gamma_{\mathfrak{m}}([M])$ has support in $[\operatorname{depth} M, \dim M]$.

Definition 1.5.17 (Cohen-Macaulay)**.** We call an R-module M **Cohen-Macaulay** if $\operatorname{depth} M = \dim M$. A ring R is Cohen-Macaulay if it is Cohen-Macaulay as an R-module.

Remark 1.5.18. Recall that a free resolution is a quasi-isomorphic representative $F_\bullet$ of $[M]$ in $D(R)$; i.e., $F_\bullet$ is a complex with each F_i free and with $[M] \cong_q F_\bullet$. The complex $F_\bullet$ need not be bounded; that is, infinite free resolutions exist. It can even be the case that every free resolution $F_\bullet \to M \to 0$ is unbounded.

Definition 1.5.19 (projective dimension)**.** We define the **projective dimension**

$$p\dim(M) = \min_n \left\{ [M] \cong_q F_\bullet \mid F_i = 0 \text{ for } i > n, F_i \text{ is projective} \right\}.$$

Theorem 1.5.20 (Auslander-Buchsbaum)**.** *Let R be noetherian. Let M be a finitely generated R-module. If M has finite projective dimension, then $p\dim M + \operatorname{depth} M = \operatorname{depth} R$.*

Remark 1.5.21. Any finitely generated module over a polynomial ring has finite projective dimension, by the Hilbert Syzygy Theorem. Hilbert's motivation was to study R^G, the ring of invariants. That is, he wished to construct $\cdots F_1 \to F_0 \to R^G \to 0$ with $\operatorname{rank} F_i < \infty$.

Definition 1.5.22 (regular sequence)**.** Let R be a noetherian ring. Let M be an R-module. A sequence $x_1, ..., x_d \in R$ is called an **M-sequence** (or **M-regular sequence**, or **regular sequence**) provided $(x_1, ..., x_d)M \neq M$, x_1 is not a zero divisor on M, and x_i is not a zero divisor on ${M}/{(x_1, ..., x_{i-1})}$ for all i.

☡ **Warning! 1.5.23.** The order of a regular sequence matters!

Example 1.5.24. Consider $R = k[x, y, z]$. The sequence $xy, xz, x-1$ is not a regular sequence on R, but $x-1, xy, xz$ is a regular sequence on R.

Remark 1.5.25. For finitely generated modules over local rings, any permutation of a regular sequence is regular.

Theorem 1.5.26 (Rees)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Let M be a finitely generated R-module. If $x_1, ..., x_d$ is an M-regular sequence of maximal length, then for each $i \in \{1, ..., d\}$,*

$$\operatorname{Ext}^i(k, M) \cong \begin{cases} \operatorname{Hom}\left(k, M/(x_1, ..., x_d)M\right) & \textit{if } i = d; \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 1.5.27. The length d of a maximal regular sequence is $\min_n\{\operatorname{Ext}^n(k, M) \neq 0\}$; i.e., if R is local and M is finitely generated, then all maximal M-regular sequences have the same length.

Definition 1.5.28 (depth 2)**.** Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Define $\operatorname{depth} M = \min_n\{\operatorname{Ext}^n(k, M) \neq 0\}$.

Remark 1.5.29. Let us confirm that our two definitions of depth agree. Let $0 \to M \to E^\bullet$ be an injective resolution. We have $\operatorname{Ext}^n(k, M) = h^n(\operatorname{Hom}(k, E^\bullet))$. For any R-module N, note that

$$\operatorname{Hom}(k, N) \cong \operatorname{Hom}\left(R/\mathfrak{m}, N\right) \cong \operatorname{Hom}(k, \Gamma_\mathfrak{m}(N)).$$

Thus $\operatorname{Hom}(k, E^\bullet) \cong \operatorname{Hom}(k, \Gamma_\mathfrak{m}(E^\bullet)) \cong \operatorname{Hom}(k, \mathbf{R}\Gamma_\mathfrak{m}(M))$. So $\operatorname{Ext}^n(k, M) \cong h^n(\operatorname{Hom}(k, \mathbf{R}\Gamma_\mathfrak{m}(M))$, and one may check that

$$h^n(\operatorname{Hom}(k, \mathbf{R}\Gamma_\mathfrak{m}(M)) \cong \begin{cases} \operatorname{Hom}(k, H_\mathfrak{m}^{\operatorname{depth} M}(M)) & \text{if } n = \operatorname{depth} M; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5.30. The above computation makes it clear that $\operatorname{depth} R \leq \dim R$. We can also redefine Cohen-Macaulay local rings:

Definition 1.5.31 (Cohen-Macaulay 2)**.** A local ring $(R, \mathfrak{m})$ is Cohen-Macaulay provided some system of parameters is a regular sequence (equivalently, all systems of parameters are regular sequences).

Remark 1.5.32. The key connection for us between local cohomology and singularities in positive characteristic is the following. The (iterated) Frobenius $F^e : R \to R$ induces a natural morphism of complexes $\mathbf{R}\Gamma_\mathfrak{a}(R) \to F^e_*\mathbf{R}\Gamma_{\mathfrak{a}^{[p^e]}}(R) \cong_q F^e_*\mathbf{R}\Gamma_\mathfrak{a}(R)$, as Frobenius powers are cofinal with ordinary powers (**Problem Set 1 #5**). This induces a Frobenius action on cohomology which we denote $\rho^e : H^i_\mathfrak{a}(R) \to F^e_* H^i_\mathfrak{a}(R)$. Explicitly, $\rho^e(r\eta) = r^{p^e}\rho(\eta)$.

Remark 1.5.33. We may view this as an additive map $\rho^e : H^i_\mathfrak{m}(R) \to H^i_\mathfrak{m}(R)$ which is not R-linear, but does satisfy $\rho^e(r\eta) = r^{p^e}\rho(\eta)$. This may be called F-semilinear, or p^e-linear.

Definition 1.5.34 (Frobenius action)**.** We say $\rho : M \to M$ is a **Frobenius action** if $\rho(rm) = r^p\rho(m)$.

Remark 1.5.35. Having such a Frobenius action $\rho : M \to M$ makes M into a left $R\{F\}$-module, where $R\{F\} = R[\chi]/(\chi r - r^p\chi)$. Note that $R\{F\}$ is a non-commutative ring. A left $R\{F\}$-module is an R-module M with a Frobenius action $\rho : M \to M$.

Remark 1.5.36. Recall that for any field k, a $k[T]$-module V is a k-vector space with a linear transformation $T : V \to V$. See the analog to (M, ρ).

Remark 1.5.37. Note that we can iterate ρ, getting $1, \rho, \rho^2, \rho^3, ...$

Example 1.5.38. In the case $\mathfrak{a} = (f_1, ..., f_s) \subseteq R$, the action on $H^s_{\mathfrak{a}}(R) = \varinjlim R/(f_1{}^m, ..., f_s{}^m)$ is $\rho([g + (f_1{}^m, ..., f_s{}^m)]) = [g^p + (f_1{}^{pm}, ..., f_s{}^{pm})]$.

☡ **Warning! 1.5.39.** Local cohomology is almost never finitely generated!

Remark 1.5.40. One way to study large modules is to consider associated primes; i.e., prime ideals $\mathfrak{p}$ for which $R/\mathfrak{p} \hookrightarrow M$. For example, one may have that

$$\bigoplus_{\mathfrak{p}\in \operatorname{Ass}(M)} R/\mathfrak{p} \hookrightarrow M,$$

and try to show that $|\operatorname{Ass}(M)| < \infty$. Unfortunately in general, $H^i_{\mathfrak{a}}(M)$ can fail to have finitely many associated primes.

Remark 1.5.41. In the local setting, that is, when $(R, \mathfrak{m}, k)$ is a local ring, $\mathfrak{m}$ is an associated prime of the local cohomology module $H^i_{\mathfrak{a}}(R)$ via $k \hookrightarrow H^i_{\mathfrak{a}}(R)$. Even when $\mathfrak{m}$ is the only associated prime, the socle, which is the largest k-vector space in $H^i_{\mathfrak{a}}(R)$, can be infinitely generated.

Definition 1.5.42 (simple module)**.** An R-module M is **simple** if $M \neq 0$ and M has no nonzero proper submodules; i.e., if $N \subsetneq M$, then $N = 0$.

Definition 1.5.43 (essential submodule)**.** Let M be an R-module. An **essential submodule** of M is a submodule N such that for every submodule H of M, $H \cap N = 0$ implies that $H = 0$. Equivalently, we say that M is an essential extension of N.

Definition 1.5.44 (socle)**.** The **socle** of a module M over a ring R is the set

$$\begin{aligned}\operatorname{soc}(M) &= \sum\{N \mid N \text{ is a simple submodule of } M\}\\ &= \bigcap\{E \mid E \text{ is an essential submodule of } M\}.\end{aligned}$$

Remark 1.5.45. If M is an artinian module, then $\operatorname{soc}(M)$ is an essential submodule of M.

Example 1.5.46. The first local cohomology with infinitely many associated primes is due to Katzman in 2002. Let $R = k[x, y, s, t, u, v]/(sx^2v^2 - (t + s)xyuv + ty^2u^2)$. Katzman identified the associated primes of $H^2_{(u,v)}(R)$ with $k[s, t]$-irreducible factors of $\sum(-1)^i(t^i + st^{i-1} + \cdots + s^{i-1}t + s^i)$. In 2004, Singh-Swanson gave examples over domains.

Theorem 1.5.47 (Huneke-Sharp)**.** *If R is a regular ring of characteristic p, then*

$$\operatorname{Ass}\left(H^i_{\mathfrak{a}}(R)\right) \subseteq \operatorname{Ass}\left(\operatorname{Ext}^i\left(R/\mathfrak{a}, R\right)\right).$$

That is, $|H^i_{\mathfrak{a}}(R)| < \infty$.

Proof. Without loss of generality, if $\mathfrak{p} \in \operatorname{Ass}(H^i_{\mathfrak{a}}(R))$, then we can assume R is local with maximal ideal $\mathfrak{p}$. That is, we can assume that the socle of $H^i_{\mathfrak{a}}(R)$ is not zero. Recall from **Remark 1.5.5** that

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, R\right).$$

Some for some e, the socle of $\operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, R\right)$ is not zero. Since R is regular, by **Theorem 1.1.24 [Kunz]**, the Frobenius is flat, so

$$\operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, R\right) \cong \operatorname{Ext}^i\left(R/\mathfrak{a}, R\right) \otimes F^e_* R.$$

Now note that $\mathfrak{p}$ in $(R, \mathfrak{p}, k)$ is an associated prime of M if and only if $\operatorname{depth} M = 0$, since

$$\operatorname{depth} M = \min_n\{H^n_{\mathfrak{p}}(M) \neq 0\}$$

and ${R}/{\mathfrak{p}} \cong k \hookrightarrow H^n_{\mathfrak{p}}(M)$ for all $n \geq 0$. (See **Remark 1.5.41**.) By **Theorem 1.5.20 [Auslander-Buchsbaum]**, $p\dim M = \operatorname{depth} R$. Thus, by flatness,

$$p\dim\left(\operatorname{Ext}^i\left({R}/{\mathfrak{a}}, R\right) \otimes_R F^e_* R\right) = p\dim\left(\operatorname{Ext}^i\left({R}/{\mathfrak{a}}, R\right)\right).$$

The result follows. □

Example 1.5.48. Consider $R = {k[x,y,z,w]}/{(xz-yw)}$ with $\mathfrak{m} = (x,y,z,w)$. In $H^2_{(x,y)}(R)$, one can check that $\mathfrak{m}$ is the only associated prime, but the socle is infinitely generated, as for each a, the element $\left[\frac{w^{a-1}y^{a-1}}{x^a y^a}\right]$ is annihilated by $\mathfrak{m}$.

1.6 Anti-nilpotent Rings

We know that in general, local cohomology is not finitely generated, nor has finitely many associated primes. What sort of finiteness can we expect in the case that R is an F-split ring?

Definition 1.6.1 (F-stable)**.** Let (M,ρ) be an $R\{F\}$-module. Call $N \subseteq M$ F**-stable** if $\rho(N) \subseteq N$. (That is, using F_*- notation, we say $\rho(N) \subseteq F_*N$.)

Example 1.6.2. If $R = k[x_1,...,x_d]_{\mathfrak{m}}$, then the only F-stable submodule of $H^d_{\mathfrak{m}}(R)$ is itself. One can easily check when $d=1$; observe that if $\eta = \left[\frac{1}{x^a}\right] \in H^1_{(x)}(k[x])$, then

$$\rho(\eta) = \rho\left(\left[\frac{1}{x^a}\right]\right) = \frac{1}{x^{pa}}.$$

If $N \subseteq H^1_{(x)}(k[x])$ and N is F-stable, then $\left\{\left[\frac{1}{x^{pa}}\right] \mid a \in \mathbf{N}\right\} \subseteq N$, and scaling by x gives the result.

Theorem 1.6.3 (Ma)**.** *If $(R,\mathfrak{m})$ is F-split, then for each i, there are only finitely many F-stable submodules of $H^i_{\mathfrak{m}}(R)$.*

Remark 1.6.4. Theorem 1.6.3 [Ma] generalizes results of Enescu-Hochster, which assumed R is a Gorenstein ring. (See **Definition 1.10.42 [Gorenstein]** to come.) We will prove **Theorem 1.6.3 [Ma]** using a result by Enescu-Hochster (**Theorem 1.6.6**) that deals with the following related definition.

Definition 1.6.5 (anti-nilpotent)**.** Let $(R,\mathfrak{m})$ be a local ring. Let (W,ρ) be an $R\{F\}$-module. Call W **anti-nilpotent** provided for each F-stable submodule V of W, ρ acts injectively on ${W}/{V}$. That is, $\rho(w) \in V$ if and only if $w \in V$.

Theorem 1.6.6 (Enescu-Hochster)**.** *If W is an anti-nilpotent $R\{F\}$-module, then it has only finitely many F-stable submodules.*

Remark 1.6.7. The proof of **Theorem 1.6.6 [Enescu-Hochster]** utilizes a category of $\mathcal{F}$-modules; i.e., $R\{F\}$-modules M with an isomorphism $\theta : M \to M \otimes_R F_*R$. Lyubeznik gave a fully faithful functor from artinian $R\{F\}$-modules which is exact on the subcategory of anti-nilpotent modules. The finiteness comes from an older result of Hochster that uses "noetherian induction." The idea is that to prove a theorem about noetherian modules, do so by contradiction. If M is a counterexample, then $\left\{N \subseteq M \mid {M}/{N} \text{ is a counterexample}\right\} \neq \emptyset$, since it contains 0. By Zorn's lemma, we pick N maximal and work with ${M}/{N}$; that is, we can assume that all proper quotients of M satisfy the theorem.

Remark 1.6.8. Such a technique can be used to prove the following:

Claim. *If R is a noetherian ring, then R has finitely many minimal primes.*

Proof. Suppose R is a noetherian ring, and assume via noetherian induction that all quotients of R have finitely many minimal primes. R cannot be a domain, so pick $x \neq 0$ and $y \neq 0$ in R such that $xy = 0$. Any minimal prime of R must contain x or y. If $x \in \mathfrak{p}$, then $\mathfrak{p}/(x)$ is a minimal prime of $R/(x)$, and symmetrically, $\mathfrak{p}/(y)$ is a minimal prime of $R/(y)$. Thus, R has finitely many minimal primes. $\square$

Remark 1.6.9. We also need the following lemmas to prove **Theorem 1.6.3** [**Ma**].

Lemma 1.6.10. *Let (W, ρ) be an $R\{F\}$-module. W is anti-nilpotent if and only if for each $\eta \in W$, $\eta \in \operatorname{span}_R\{\rho(\eta), \rho^2(\eta), \rho^3(\eta), ...\}$.*

Proof. Let W be anti-nilpotent. The submodule $V = \operatorname{span}_R\{\rho(\eta), \rho^2(\eta), ...\}$ is F-stable, and clearly $\rho(\eta) \in V$. As W is anti-nilpotent, ρ acts injectively on W/V, so $\rho(\eta) \in V$ (that is, $\rho(\eta) = 0 \in W/V$) implies $\eta \in V$ (that is, $\eta = 0$ in W/V).

Conversely, let $V \subseteq W$ be any F-stable submodule of W. Proving the contrapositive, suppose W is not anti-nilpotent, i.e., ρ does not act injectively on W/V. Pick $\eta \notin V$ such that $\rho(\eta) \in V$. So $\operatorname{span}_R\{\rho(\eta), \rho^2(\eta), ...\} \subseteq V$, but $\eta \notin \operatorname{span}_R\{\rho(\eta), \rho^2(\eta), ...\}$. $\square$

Lemma 1.6.11. *Let R be an F-split ring. Let $\eta \in H^i_{\mathfrak{m}}(R)$. Let $N \subseteq H^i_{\mathfrak{m}}(R)$. If $F_*\eta$ is in the F_*R-span of the image of N under $H^i_{\mathfrak{m}}(R) \to F_*H^i_{\mathfrak{m}}(R)$, then $\eta \in N$.*

Proof. We prove an actually strong result. Consider the following:

Claim. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local rings. Suppose there exists a split injection $R \hookrightarrow S$ with $\sqrt{\mathfrak{m}S} = \mathfrak{n}$. Let $N \subseteq H^i_{\mathfrak{m}}(R)$. If η is in the S-span of the image of N in $H^i_{\mathfrak{n}}(S)$ under the map induced by the injection, then $\eta \in N$.*

Proof. Denote the splitting by $\gamma : S \to R$. There is a natural map

$$\varphi : S \otimes_R H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}S}(S) \cong H^i_{\mathfrak{n}}(S)$$

arising from tensoring the Čech complex computing $H^i_{\mathfrak{m}}(R)$ by S. This gives a natural diagram

$$\begin{tikzcd}
H^i_{\mathfrak{n}}(S) & & S \otimes_R H^i_{\mathfrak{m}}(R) \arrow[ll, "\varphi"'] \\
& H^i_{\mathfrak{m}}(R) \arrow[ul, "j_1"] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

where $j_2(\eta) = 1 \otimes \eta$ and j_1 is induced by $R \hookrightarrow S$. The splitting γ induces a map

$$q_1 : H^i_{\mathfrak{n}}(S) \to H^i_{\mathfrak{n}}(R) \cong H^i_{\mathfrak{m}}(R)$$

coming from $S \cong R \oplus P$ as R-modules, so indeed we have

$$q_1 : H^i_{\mathfrak{n}}(S) \to H^i_{\mathfrak{n}}(R \oplus P) \cong H^i_{\mathfrak{n}}(R) \oplus H^i_{\mathfrak{n}}(P) \xrightarrow{proj} H^i_{\mathfrak{n}}(R) \cong H^i_{\mathfrak{m}}(R).$$

We also have a map $q_2 : S \otimes_R H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ defined by $q_2(s \otimes \eta) = \gamma(s) \cdot \eta$ extended linearly. Note that both $q_1 j_1 = \mathrm{id}$ and that the following diagram commutes:

$$\begin{tikzcd}
H^i_{\mathfrak{n}}(S) \arrow[dr, "q_1"'] & & S \otimes_R H^i_{\mathfrak{m}}(R) \arrow[ll, "\varphi"'] \arrow[dl, "q_2"] \\
& H^i_{\mathfrak{m}}(R) &
\end{tikzcd}$$

Now assume that η is in the S-span of the image of N in $H^i_{\mathfrak{n}}(S)$. That is, $j_1(\eta) = \sum_k s_k j_1(n_k)$ for $n_k \in N$ and $s_k \in S$. Using the above, we have

$$\begin{aligned}
\eta &= q_1(j_1(\eta)) \\
&= q_1\left(\sum_k s_k j_1(n_k)\right) \\
&= q_1\left(\sum_k s_k \varphi(j_2(n_k))\right) \\
&= \sum_k q_1(s_k\varphi(j_2(n_k))) \\
&= \sum_k q_1(\varphi(s_k j_2(n_k))) \\
&= \sum_k q_2(s_k j_2(n_k)) \\
&= \sum_k q_2(s_k(1 \otimes n_k)) \\
&= \sum_k q_2(s_k \otimes n_k) \\
&= \sum_k \gamma(s_k) n_k,
\end{aligned}$$

which is in N, as desired. $\square$

The lemma follows by setting $S = F_*R$. $\square$

*Proof of **Theorem 1.6.3 [Ma]**.* Let $(R, \mathfrak{m})$ be F-split. We need to show that there are only finitely many F-stable submodules of $H^i_{\mathfrak{m}}(R)$. By **Theorem 1.6.6 [Enescu-Hochster]**, it is enough to show that F-split implies anti-nilpotent. By **Lemma 1.6.10**, it is enough to check that for each element $\eta \in H^i_{\mathfrak{m}}(R)$, $\eta \in \operatorname{span}_R\{\rho(\eta), \rho^2(\eta), ...\}$, for then $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent.

First, for $j > 0$, set $N_j = \operatorname{span}_R\{\rho^k(\eta) \mid k \geq j\}$. Note that $H^i_{\mathfrak{m}}(R) = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$. Since $H^i_{\mathfrak{m}}(R)$ is artinian (**Remark 1.5.5**), we can pick e minimal so that $N_e = N_{e+1}$. The result follows if $e = 0$.

Assume for the sake of contradiction that $e > 0$, so $\rho^{e-1}(\eta) \notin N_e$. By **Lemma 1.6.11**, $\rho^{e-1}(\eta)$ is not in the F_*R-span of the image of $N_e = N_{e+1}$ under $H^i_{\mathfrak{m}}(R) \xrightarrow{\rho} F_*H^i_{\mathfrak{m}}(R)$. But clearly $\rho(\rho^{e-1}(\eta)) = \rho^e(\eta) \in N_e$, so we have a contradiction. $\square$

☡ **Warning! 1.6.12.** The converse of **Theorem 1.6.3 [Ma]** is false! Local cohomology $H^i_{\mathfrak{m}}(R)$ having finitely many F-stable submodules need not imply that R is F-split. Consider $R = {}^{k[x,y,z]}\big/_{(x^3+y^3+z^3)}$. One can check that the only F-stable submodule of $H^2_{\mathfrak{m}}(R)$ is the socle, but R is F-split only whenever $p \equiv 1 \bmod 3$, by **Corollary 1.4.24 [Fedder's Criterion]**

Theorem 1.6.13 (Schwede-Tucker)**.** *Let $(R, \mathfrak{m})$ be an F-split local ring. Let the embedding dimension of R be ν. The number of primes $\mathfrak{p}$ compatible with the splitting and of dimension d (that is, $\dim {}^{R}\!/_{\mathfrak{p}} = d$) is at most $\binom{\nu}{d}$.*

Remark 1.6.14. The bound $\binom{\nu}{d}$ comes from the following theorem:

Theorem. *If S is set of primes such that the set of finite intersections,*

$$\left\{ \bigcap_{\mathfrak{p}_i \in T} \mathfrak{p}_i \mid T \subseteq S, |T| < \infty \right\},$$

is closed under sums, then the number of primes in S of dimension d is at most $\binom{\nu}{d}$.

1.7 F-injective Rings

Lemma 1.7.1. *If $(R, \mathfrak{m})$ is F-split, then for each $\mathfrak{a} \subseteq R$, the natural Frobenius action on $H^i_{\mathfrak{a}}(R)$ is injective.*

Proof. Set $\varphi : F_*R \to R$ the splitting, so

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & F_*R \arrow[r, "\varphi"] & R \end{tikzcd}$$

Apply $H^i_{\mathfrak{a}}-$ to get

$$\begin{tikzcd} H^i_{\mathfrak{a}}(R) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & F_*H^i_{\mathfrak{a}}(R) \arrow[r] & H^i_{\mathfrak{a}}(R) \end{tikzcd}$$

Hence, $H^i_{\mathfrak{a}}(R) \to F_*H^i_{\mathfrak{a}}(R)$ is injective, as desired. $\square$

Definition 1.7.2 (F-injective)**.** Call a local ring $(R, \mathfrak{m})$ **F-injective** provided $H^i_{\mathfrak{m}}(R) \to F_*H^i_{\mathfrak{m}}(R)$ is injective for all i.

Remark 1.7.3. By **Lemma 1.7.1** above, F-split implies F-injective. The converse fails.

Example 1.7.4. Let $R = {}^{k[x,y,z,w]_{\mathfrak{m}}}\!\big/_{(xy, xz, y(z-w^2))}$. One can check that R is not F-split using **Corollary 1.4.24 [Fedder's Criterion]** in `Macaulay2`. That is, check that

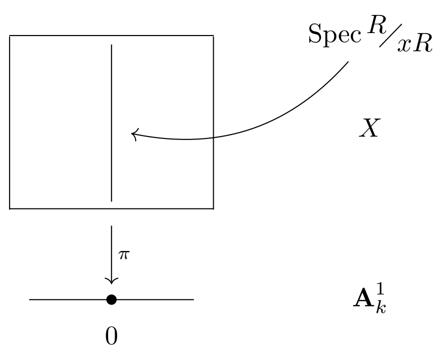
$$\Big((xy, xz, y(z-w^2))^{[p]} : (xy, xz, y(z-w^2))\Big) \subseteq \mathfrak{m}^{[p]}.$$

How do we show that R is F-injective, though?

Remark 1.7.5. One technique to show that a ring is F-injective is through "deformation."

Definition 1.7.6 (deform)**.** A property P **deforms** for a local ring $(R, \mathfrak{m})$ provided for each regular element $x \in \mathfrak{m}$, if ${}^{R}\!/_{xR}$ has property P, then R has property P.

Remark 1.7.7. The visual suggestion of deformation is clearer when we consider the geometric perspective. Let $X = \operatorname{Spec} R$. Map $k[t] \to R$ by $t \mapsto x$ a regular element. We get a map of schemes $X \xrightarrow{\pi} \mathbf{A}^1_k = \operatorname{Spec} k[t]$. The map $\operatorname{Spec} {}^{R}\!/_{xR} \hookrightarrow X$ is the fiber over 0.



Remark 1.7.8. Note that $P =$ "being regular" does not deform. Consider $R = {}^{k[x,y,z]}\!\big/_{z(y^2 - x^3)}$. R is not regular, but ${}^{R}\!/_{zR} \cong k[x, y]$ is.

Remark 1.7.9. Property $P =$ "being Cohen-Macaulay" does deform. See **Problem Set 3 #4**.

Remark 1.7.10. A natural question thus arises: does F-injective deform for all local rings $(R, \mathfrak{m})$? In fact, this is open in general! There are some partial results, however.

Theorem 1.7.11 (Fedder). *If $(R,\mathfrak{m})$ is Cohen-Macaulay, then F-injective deforms.*

Proof. Let $x \in \mathfrak{m}$ be a regular element. Let ${}^{R}\!/\!_{xR}$ be F-injective. We need to show that R is F-injective; i.e., we must show that $H^d_{\mathfrak{m}}(R) \xrightarrow{\rho} H^d_{\mathfrak{m}}(R)$ is injective. This suffices, as R is Cohen-Macaulay, so all other local cohomology modules are 0. Recall that we may pick a system of parameters $x_1, ..., x_d$ with $x = x_1$, $\sqrt{(x_1, ..., x_d)} = \mathfrak{m}$, and d minimal. An element $\eta \in H^d_{\mathfrak{m}}(R)$ is a class

$$\eta = \left[\frac{f}{{x_1}^{a_1} \cdots {x_d}^{a_d}}\right],$$

and we may pick a representative for η with a_1 minimal.

We know $\eta = 0$ if and only if there exists k such that $f(x_1 \cdots x_d)^k \in ({x_1}^{a_1+k}, ..., {x_d}^{a_d+k})$, by **Lemma 1.5.10**. As R is Cohen-Macaulay, by **Definition 1.5.31 [Cohen-Macaulay 2]**, $x_1, ..., x_d$ is a regular sequence, so $\eta = 0$ if and only if $f \in ({x_1}^{a_1}, ..., {x_d}^{a_d})$; i.e., $k = 0$. Also,

$$\rho(\eta) = \left[\frac{f^p}{{x_1}^{pa_1} \cdots {x_d}^{pa_d}}\right]$$

Set $\overline{\eta}$ for the image of η in $H^i_{\mathfrak{m}}\left({}^{R}\!/\!_{xR}\right)$. Assume $\rho(\eta) = 0$, so

$$\left[\frac{f^p}{{x_1}^{pa_1} \cdots {x_d}^{pa_d}}\right] = 0,$$

which implies $f^p \in ({x_1}^{pa_1}, ..., {x_d}^{pa_d})$ in R. Thus $f^p \in ({x_2}^{pa_2}, ..., {x_d}^{pa_d})$ in ${}^{R}\!/\!_{xR}$. That is,

$$\rho\left(\left[\frac{f}{{x_2}^{a_d} \cdots {x_d}^{a_d}}\right]\right) = \rho\left(\overline{\eta}\right) = 0$$

in ${}^{R}\!/\!_{xR}$. As ${}^{R}\!/\!_{xR}$ is F-injective, $\overline{\eta} = 0$ in ${}^{R}\!/\!_{xR}$. Thus, $f \in ({x_2}^{a_2}, ..., {x_d}^{a_d})$ in ${}^{R}\!/\!_{xR}$, and hence we have $f \in (x_1, {x_2}^{a_2}, ..., {x_d}^{a_d})$ in R. We can write

$$f = r_1 x_1 + \sum_{i=2}^{d} r_i {x_i}^{a_i};$$

we get

$$\left[\frac{f}{{x_1}^{a_1} \cdots {x_d}^{a_d}}\right] = \left[\frac{r_1 x_1}{{x_1}^{a_1} \cdots {x_d}^{a_d}}\right] + \sum_{i=2}^{d} \left[\frac{r_i {x_i}^{a_i}}{{x_1}^{a_1} \cdots {x_d}^{a_d}}\right],$$

and

$$\sum_{i=2}^{d} \left[\frac{r_i {x_i}^{a_i}}{{x_1}^{a_1} \cdots {x_d}^{a_d}}\right] = 0$$

in $H^i_{\mathfrak{m}}(R)$ by a Čech complex computation. This contradicts the minimality of a_1, unless $\eta = 0$, which we needed to show. ☐

Remark 1.7.12. We can provide an alternative proof for **Theorem 1.7.11** as follows:

Proof. Consider the short exact sequence

$$0 \to R \xrightarrow{\cdot x} R \to {}^{R}\!/\!_{xR} \to 0.$$

We adjust the Frobenius action to get a map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p-1}F^e"] & R \arrow[r] \arrow[d, "F^e"] & R/xR \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "\cdot x"'] & F^e_* R \arrow[r] & F^e_* R/xR \arrow[r] & 0
\end{tikzcd}$$

This induces in local cohomology, for $d = \dim R$, the following:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, two heads] & H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \arrow[r] \arrow[d, hook, "F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p-1}F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & F^e_* H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \arrow[r] & F^e_* H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"'] & F^e_* H^d_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

We now assert the following:

Claim. *$x^{p-1}F^e$ is injective in the local cohomology diagram above.*

Proof. We use the fact that if $\mathrm{soc}(H^d_{\mathfrak{m}}(R)) \cap \ker(x^{p-1}F^e) = 0$, then $\ker(x^{p-1}F^e) = 0$; that is, that $\mathrm{soc}(H^d_{\mathfrak{m}}(R)) \hookrightarrow H^d_{\mathfrak{m}}(R)$ is an essential extension. This holds by **Remark 1.5.45**, as $H^d_{\mathfrak{m}}(R)$ is artinian by **Remark 1.5.5**. But indeed, we can show so explicitly:

Conflate $F^e_* N$ with N for all objects N. Set $\eta \in \mathrm{soc}(H^d_{\mathfrak{m}}(R)) \cap \ker(x^{p-1}F^e)$, so $x \cdot \eta = 0$. That is, η lifts to $H^{d-1}_{\mathfrak{m}}\left(R/xR\right)$ in the local cohomology diagram above. Now, as $H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \hookrightarrow H^d_{\mathfrak{m}}(R)$, we see that $\eta = 0$. Now, apply the 5 Lemma. $\square$

$\square$

Example 1.7.13. Let $R = k[x,y,z,w]_{\mathfrak{m}}/(xy, xz, y(z-w^2))$ as in **Example 1.7.4**. R is Cohen-Macaulay, and $w \in \mathfrak{m}$ is a regular element. The ring R/wR is

$$R/wR \cong k[x,y,z,w]_{\mathfrak{m}}/(xy, xz, yz).$$

One can check that R/wR is F-split using **Corollary 1.4.24 [Fedder's Criterion]**, hence F-injective by **Lemma 1.7.1**. Thus R is F-injective by **Theorem 1.7.11**.

Remark 1.7.14. Cohen-Macaulay is not the only condition that gives a partial result to the question of F-injectivity deforming.

Definition 1.7.15 (surjective element)**.** For a local ring $(R, \mathfrak{m})$, call $x \in \mathfrak{m}$ a **surjective element** if x is regular and for all $\ell \geq 0$, $H^i_{\mathfrak{m}}\left(R/x^\ell R\right) \to H^i_{\mathfrak{m}}\left(R/xR\right)$ is surjective.

Theorem 1.7.16 (Horiuchi-Miller-Shimomoto)**.** *If $(R, \mathfrak{m})$ is a local ring, $x \in \mathfrak{m}$ is a surjective element, and R/xR is F-injective, then R is F-injective.*

Remark 1.7.17. Property P = "being F-split" does not deform, by **Theorem 1.7.18**. However, there is a deformation result involving F-split and F-injective; see **Theorem 1.7.19**.

Theorem 1.7.18 (Singh)**.** *Let $m, n \in \boldsymbol{Z}$ such that $m - \frac{m}{n} > 2$ and*

$$R = k[A,B,C,D,T]/I$$

where $I = I_2 \begin{bmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{bmatrix}$. If $\gcd(p, m) = 1$, then R/TR is F-split, but R is not F-split.

Theorem 1.7.19. *If $(R, \mathfrak{m})$ is local, $x \in \mathfrak{m}$ is regular, and ${}^{R}\!\diagup_{xR}$ is F-split, then R is F-injective.*

Proof. It suffices to show x is a surjective element, by **Theorem 1.7.16** [**Horiuchi-Miller-Shimomoto**]. Set $\ell > 0$, and let $C = \operatorname{coker}\left(\Phi : H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \to H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)\right)$. We have a diagram

$$\begin{tikzcd}
H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \arrow[r, "\Phi"] \arrow[d, "\rho"] & H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \arrow[r] \arrow[d, "\rho"] & C \arrow[r] \arrow[d, "\rho_C"] & 0 \\
F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \arrow[r, "F^e_*\Phi"] & F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \arrow[r] & F^e_* C \arrow[r] & 0
\end{tikzcd}$$

where all ρ are Frobenius actions. The image of $H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right)$ in $H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)$ is F-stable, by checking **Definition 1.6.1** [***F*-stable**] using the diagram:

$$\rho\left(\Phi\left(H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell}\right)\right)\right) = F^e_*\Phi\left(\rho\left(H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right)\right)\right).$$

As ${}^{R}\!\diagup_{xR}$ is F-split, all local cohomology of ${}^{R}\!\diagup_{xR}$ is anti-nilpotent. Thus, ρ_C is injective.

For $e \gg 0$, the map $\rho : H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)$ factors as

$$H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^{p^e} R}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)$$

Denote by φ the piece of the above map $\varphi : H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^{p^e} R}\right) \to F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right)$.

So a diagram chase yields $C = 0$. Indeed, let $z \in C$. Lift z to z' in $H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)$. Map z' to $\rho(z') = \widetilde{z}$, and as ρ factors, pick $\widehat{z} \in F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right)$ such that $\varphi(z') = \widehat{z}$. See that $\widehat{z} \mapsto 0 \in F^e_* C$, and since ρ_C is injective, $C = 0$, as desired.

$$\begin{tikzcd}
 & z' \arrow[r, mapsto] \arrow[dddl] \arrow[ddd, "\rho"] & z & \\
H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \arrow[r] \arrow[d, "\rho"] & H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \arrow[r] \arrow[dl, "\varphi"] & C \arrow[r] \arrow[d, hook, "\rho_C"] & 0 \\
F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{x^\ell R}\right) \arrow[r] & F^e_* H^i_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \arrow[r] & F^e_* C \arrow[r] & 0 \\
\widehat{z} \arrow[r, mapsto] & \widetilde{z} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

Thus, x is a surjective element, and the proof is complete. $\square$

Theorem 1.7.20 (Ma-Quy)**.** *Anti-nilpotent deforms.*

Proof sketch. The proof first establishes that if ${}^{R}\!\diagup_{xR}$ is anti-nilpotent, then x is a surjective element. Once done, let $N \subseteq H^i_{\mathfrak{m}}(M)$ be an F-stable submodule. One checks that

$$L = \bigcap_{t \in \mathbf{N}} x^t N$$

is also F-stable, then for each e, lets $\delta : H^{i-1}_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right) \to H^i_{\mathfrak{m}}(R)$ and produces a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)\diagup_{\delta^{-1}(L)} \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R)\diagup_{L} \arrow[r] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R)\diagup_{L} \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}\left({}^{R}\!\diagup_{xR}\right)\diagup_{\delta^{-1}L} \arrow[r] & H^i_{\mathfrak{m}}(R)\diagup_{L} \arrow[r] & H^i_{\mathfrak{m}}(R)\diagup_{L} \arrow[r] & 0
\end{tikzcd}$$

Next, use a socle argument to show that $x^{p^e-1}F^e$ is injective. This forces F^e to be injective. Finally, use an argument similar to the proof of **Theorem 1.6.3 [Ma]** to the sequence

$$N \supseteq xN \supseteq x^2N \supseteq \cdots$$

to show that $N = L$. $\square$

Theorem 1.7.21 (Schwede). *Let $(R,\mathfrak{m})$ be a local ring with ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = (0)$ and $\dim R = \dim \left(R/\mathfrak{a}_1\right) = \dim \left(R/\mathfrak{a}_2\right)$. If $R/\mathfrak{a}_1$ and $R/\mathfrak{a}_2$ are Cohen-Macaulay and $R/\mathfrak{a}_1$, $R/\mathfrak{a}_2$, and $R/(\mathfrak{a}_1+\mathfrak{a}_2)$ are F-injective, then R is F-injective.*

Proof. Set $d = \dim R$, and use

$$0 \to R \to R/\mathfrak{a}_1 \oplus R/\mathfrak{a}_2 \to R/(\mathfrak{a}_1+\mathfrak{a}_2) \to 0.$$

Apply $\mathbf{R}\Gamma_{\mathfrak{m}}$ to get

$$0 \to H^{i-1}_{\mathfrak{m}}\left(R/(\mathfrak{a}_1+\mathfrak{a}_2)\right) \xrightarrow{\sim} H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right) \cong 0$$

for $i < d$. This will show that the natural Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for $i < d$.

We have also

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}\left(R/(\mathfrak{a}_1+\mathfrak{a}_2)\right) \arrow[r] \arrow[d, hook, "\rho_L"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "\rho_M"] & H^d_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^d_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right) \arrow[d, hook, "\rho_R"] \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}\left(R/(\mathfrak{a}_1+\mathfrak{a}_2)\right) \arrow[r, hook] & H^d_{\mathfrak{m}}(R) \arrow[r] & H^d_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^d_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right)
\end{tikzcd}$$

Set $\eta \in H^d_{\mathfrak{m}}(R)$ with $\rho_M(\eta) = 0$. Perform a diagram chase to see that $\eta = 0$. $\square$

Theorem 1.7.22 (Quy-Shimomoto). *Let $(R,\mathfrak{m})$ be a local ring with ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$. If $R/\mathfrak{a}_1$ and $R/\mathfrak{a}_2$ are F-injective and $R/(\mathfrak{a}_1+\mathfrak{a}_2)$ is anti-nilpotent, then $R/(\mathfrak{a}_1\cap\mathfrak{a}_2)$ is F-injective.*

Proof. Consider the short exact sequence

$$0 \to R/(\mathfrak{a}_1\cap\mathfrak{a}_2) \to R/\mathfrak{a}_1 \oplus R/\mathfrak{a}_2 \to R/(\mathfrak{a}_1+\mathfrak{a}_2) \to 0.$$

The long exact sequence becomes

$$\cdots \to H^{i-1}_{\mathfrak{m}}\left(R/(\mathfrak{a}_1+\mathfrak{a}_2)\right) \xrightarrow{\delta} H^i_{\mathfrak{m}}\left(R/(\mathfrak{a}_1\cap\mathfrak{a}_2)\right) \to H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right) \to \cdots$$

Use this to write the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\delta \arrow[r] \arrow[d, "\rho"] & H^i_{\mathfrak{m}}\left(R/(\mathfrak{a}_1\cap\mathfrak{a}_2)\right) \arrow[r] \arrow[d] & H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right) \arrow[d] \\
0 \arrow[r] & \operatorname{im}\delta \arrow[r] & H^i_{\mathfrak{m}}\left(R/(\mathfrak{a}_1\cap\mathfrak{a}_2)\right) \arrow[r] & H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right)
\end{tikzcd}$$

The key observation is that by the first isomorphism theorem,

$$\operatorname{im}\delta \cong H^i_{\mathfrak{m}}\left(R/(\mathfrak{a}_1\cap\mathfrak{a}_2)\right)/\ker\delta,$$

and note that $\ker\delta$ is F-stable. As $R/(\mathfrak{a}_1\cap\mathfrak{a}_2)$ is anti-nilpotent by hypothesis, ρ is injective, and the proof follows by a diagram chase. $\square$

Remark 1.7.23. One might wonder what the hypotheses of **Theorem 1.7.21 [Schwede]** and **Theorem 1.7.22 [Quy-Shimomoto]** are actually requiring, perhaps in a geometric sense. If we set $X = \operatorname{Spec} R$, then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ define subschemes of X. Let $Y_1 = \operatorname{Spec} R/\mathfrak{a}_1$ and $Y_2 = \operatorname{Spec} R/\mathfrak{a}_2$ be these subschemes, or respectively $Y_1 = V(\mathfrak{a}_1)$ and $Y_2 = V(\mathfrak{a}_2)$. Recall that $V(\mathfrak{a}_1+\mathfrak{a}_2) = V(\mathfrak{a}_1)\cap V(\mathfrak{a}_2)$, $V(\mathfrak{a}_1\cap\mathfrak{a}_2) = V(\mathfrak{a}_1)\cup V(\mathfrak{a}_2)$, and $V(0) = X$.

Definition 1.7.24 (F-injective 2)**.** Call a scheme X **F-injective** if all local rings are F-injective.

Definition 1.7.25 (Cohen-Macaulay 3)**.** Call a scheme X **Cohen-Macaulay** if all local rings are Cohen-Macaulay.

Corollary 1.7.26. *If X is a reduced scheme which is a union of two subschemes Y_1 and Y_2 with $\dim X = \dim Y_1 = \dim Y_2$, Y_1 and Y_2 are Cohen-Macaulay schemes, and Y_1, Y_2, and $Y_1 \cap Y_2$ are F-injective schemes, then X is an F-injective scheme.*

Remark 1.7.27. This gives another proof that $R = k[x,y,z,w]_{\mathfrak{m}}/(xy, xz, y(z-w^2))$ is F-injective. (See **Example 1.7.13**.) Indeed,

$$\begin{aligned}(xy, xz, y(z-w^2)) &= ((x,y)\cap(z,y))\cap(x, z-w^2)\\ &= (y, xz)\cap(x, z-w^2)\\ &= \mathfrak{a}_1\cap\mathfrak{a}_2,\end{aligned}$$

and $\mathfrak{a}_1+\mathfrak{a}_2 = (y, x, z-w^2)$. Each hypothesis of **Corollary 1.7.26** can be checked to see that R is F-injective.

1.8 F-rational Rings

Definition 1.8.1 (F-rational)**.** A local ring $(R,\mathfrak{m})$ of dimension d is **F-rational** provided both

1. R is Cohen-Macaulay; i.e., $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$, and
2. $H^d_{\mathfrak{m}}(R)$ is simple as an $R\{F\}$-module; i.e., there are no proper F-stable submodules of $H^d_{\mathfrak{m}}(R)$.

Example 1.8.2. $R = k[x_1,...,x_d]_{\mathfrak{m}}$ is F-rational. We saw in **Example 1.6.2** that the only F-stable submodule of $H^d_{(x_1,\ldots,x_d)}(R)$ is itself.

Example 1.8.3. If $R = k[x,y,z]_{\mathfrak{m}}/(x^3+y^3+z^3)$, then the socle of $H^2_{\mathfrak{m}}(R)$ is F-stable and proper. Hence, R is not F-rational.

Theorem 1.8.4. *If R is an F-rational local ring, then R is F-injective.*

Proof. As R is Cohen-Macaulay, we only need to check that the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective. Note that $\left[\frac{1}{x_1\cdots x_d}\right] \neq 0$ in $H^d_{\mathfrak{m}}(R)$, as otherwise there is a k such that $(x_1\cdots x_d)^k \in ({x_1}^{k+1},...,{x_d}^{k+1})$ by **Lemma 1.5.10**. This violates the Monomial Conjecture (now a Theorem).

This also shows that $\rho\left(\left[\frac{1}{x_1\cdots x_d}\right]\right) = \left[\frac{1}{{x_1}^p\cdots {x_d}^p}\right] \neq 0$; i.e., $\ker\rho \neq H^d_{\mathfrak{m}}(R)$. Recall that $\ker\rho$ is F-stable. As R is F-rational, $\ker\rho = 0$, so R is F-injective, as desired. $\square$

Remark 1.8.5. The converse fails. Let $R = k[x,y,z]_{\mathfrak{m}}/(x^3+y^3+z^3)$. By ≳ **Warning! 1.6.12**, R is F-split when $p \equiv 1 \bmod 3$, and by **Lemma 1.7.1**, in that case, R is F-injective. However, by **Example 1.8.3**, R is not F-rational.

Remark 1.8.6. Thus far, we have the following diagram of implications:

$$\begin{tikzcd}
 & & F\text{-rational} \arrow[d, Rightarrow, "\textbf{Theorem 1.8.4}"] \\
F\text{-split} \arrow[rr, Rightarrow, "\textbf{Lemma 1.7.1}"] \arrow[dr, Rightarrow, "\textit{Proof of Theorem 1.6.3 [Ma]}"'] & & F\text{-injective} \\
 & \text{anti-nilpotent} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

None of the implications above can be reversed, but we haven't yet seen an F-injective and not anti-nilpotent ring.

Example 1.8.7 (Enescu-Hochster)**.** Let k be an infinite perfect field of characteristic $p > 2$. Let $K = k(u, v)$. Let $L = {}^{K[y]}\!\diagup_{(y^{2p} - uy^p - v)}$. Let $R = K + xL[\![x]\!] \subseteq L[\![x]\!]$. R is a complete, one-dimensional domain. We will see that R if F-injective, but R is not anti-nilpotent.

Indeed, one first uses field theory to check that ${}^{L}\!\diagup_{K}$ has infinitely man F-stable K-subspaces. Next, consider the short exact sequence

$$0 \to R \to L[\![x]\!] \to {}^{L}\!\diagup_{K} \to 0.$$

This induces in the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & {}^{L}\!\diagup_{K} \arrow[r] \arrow[d, phantom, "\cong" description, sloped] & H^1_{\mathfrak{m}}(R) \arrow[r] & H^1_{\mathfrak{m}}(L[\![x]\!]) \arrow[r] & 0 \\ & H^0_{\mathfrak{m}}\left({}^{L}\!\diagup_{K}\right) & & & \end{tikzcd}$$

This embeds ${}^{L}\!\diagup_{K}$ into the socle of $H^1_{\mathfrak{m}}(R)$; i.e., this promotes the F-stable K-subspaces of ${}^{L}\!\diagup_{K}$ to F-stable R-submodules of $H^1_{\mathfrak{m}}(R)$. Therefore, R is not anti-nilpotent, by **Theorem 1.6.6 [Enescu-Hochster]**.

On the other hand, both Frobenius actions on ${}^{L}\!\diagup_{K}$ and $H^1_{\mathfrak{m}}(L[\![x]\!])$ are injective. This lets us conclude that R is F-injective.

Definition 1.8.8 (nilpotent)**.** A map $f : A \to B$ is **nilpotent** if for each $a \in A$, there exists $e \gg 0$ such that $f^e(a) = 0$.

Remark 1.8.9. Even though F-rational implies F-injective by **Theorem 1.8.4** but F-injective does not imply anti-nilpotent by **Remark 1.8.5**, we can reverse implications if we add another hypothesis. If $(R, \mathfrak{m})$ is an F-injective local ring, then by definition $\rho : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for $i < \dim R$. For $(R, \mathfrak{m})$ further to be F-rational, we need to conclude that $H^i_{\mathfrak{m}}(R) = 0$. Note that if ρ is nilpotent and injective, then $H^i_{\mathfrak{m}}(R) = 0$.

Note, however, that the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is never nilpotent.

Theorem 1.8.10 (Smith)**.** *For an excellent local domain $(R, \mathfrak{m})$ of dimension d, the largest F-stable submodule of $H^d_{\mathfrak{m}}(R)$ is*

$$0^*_{H^d_{\mathfrak{m}}(R)} = \left\{\eta \in H^d_{\mathfrak{m}}(R) \mid \text{ there exists } c \neq 0 \text{ such that } c\rho^e(\eta) = 0 \text{ for } e \gg 0\right\},$$

which is clearly F-stable.

Corollary 1.8.11. *We say a domain R is F-rational if and only if R is Cohen-Macaulay and $0^*_{H^d_{\mathfrak{m}}} = 0$.*

Definition 1.8.12 (Srinivas-Takagi, F-nilpotent)**.** A local domain $(R, \mathfrak{m})$ of dimension d is called **F-nilpotent** provided the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is nilpotent for $i < \dim R$, and on $0^*_{H^d_{\mathfrak{m}}(R)}$ is nilpotent.

Remark 1.8.13. This condition was also studied earlier by Blickle, et. al. F-nilpotence is deeply related to various "Hodge theoretic" conditions in characteristic 0.

Lemma 1.8.14. *Let (M, ρ) be an $R\{F\}$-module. If ρ is injective and nilpotent, then $M = 0$.*

Proof. Let $\eta \in M$. By hypothesis, $\rho^e(\eta) = 0$ for $e \gg 0$. As ρ is injective, $\eta = 0$. ☐

Theorem 1.8.15 (Srinivas-Takagi)**.** *A local domain $(R, \mathfrak{m})$ is F-rational if and only if it is F-injective and F-nilpotent.*

Proof. Set $(M, \rho) = H^i_{\mathfrak{m}}(R)$ for $i < \dim R$ or $0^*_{H^d_{\mathfrak{m}}(R)}$. So $(M, \rho) = 0$ if and only if ρ is injective and nilpotent. ☐

Remark 1.8.16. We may now supplement **Remark 1.8.6**:

$$\begin{tikzcd}
 & & F\text{-rational} \arrow[d, Rightarrow, bend right, "\textbf{Theorem 1.8.4}"'] \\
F\text{-split} \arrow[rr, Rightarrow, "\textbf{Lemma 1.7.1}"] \arrow[dr, Rightarrow, "\textit{Proof of \textbf{Theorem 1.6.3} [Ma]}"'] & & F\text{-injective} \arrow[u, Rightarrow, bend right, "+F\text{-nilpotent, }\textbf{Theorem 1.8.15 [Srinivas-Takagi]}"'] \\
 & \text{anti-nilpotent} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Remark 1.8.17. Since we have discussed the techniques of deformation and gluing, one may ask: what happens for anti-nilpotent, F-rational, and F-nilpotent rings?

Any subquotient of an anti-nilpotent $R\{F\}$-module (M, ρ) is also anti-nilpotent. This forces anti-nilpotent singularities to glue.

Also, anti-nilpotent deforms. Both gluing and deformation come from [Quy-Shimomoto].

Theorem 1.8.18. *F-rational deforms. That is, let $(R, \mathfrak{m})$ be a local ring with $x \in \mathfrak{m}$ regular. If R/xR is F-rational, then R is F-rational.*

Proof. Note that if R/xR is F-rational, then R/xR is Cohen-Macaulay and F-injective. Cohen-Macaulay deforms by **Problem Set 3 #4**, and R being Cohen-Macaulay implies F-injective deforms by **Theorem 1.7.11**. In fact, since R is Cohen-Macaulay and F-injective, we have

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \arrow[r] \arrow[d, "\rho^e"] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}\rho^e"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "\rho^e"] & 0 \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

and $x^{p^e-1}\rho^e$ is injective for all $e \gg 0$. Set $N \subseteq H^d_{\mathfrak{m}}(R)$ to be F-stable. Consider

$$N \supseteq xN \supseteq x^2N \supseteq x^3N \supseteq \cdots,$$

which stabilizes, as $H^d_{\mathfrak{m}}(R)$ is artinian, to

$$L = \bigcap_{t \in \mathbf{W}} x^t N.$$

Note $L = xL$. If $L = 0$, then $x^{p^e-1}\rho^e(N) \subseteq x^{p^e-1}N = L = 0$ for $e \gg 0$. However, $x^{p^e-1}\rho^e$ is injective for $e \gg 0$, so $N = 0$.

On the other hand, we want a contradiction if $L \neq 0$. Suppose so. First, consider the following claim:

Claim. *$L \cap H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \neq 0$ in $H^d_{\mathfrak{m}}(R)$.*

Proof. Warning: one or both of these proofs is wrong. Write $L = x^tN \neq x^{t-1}N$. Pick $\eta'' \in x^{t-1}N$; then $\eta' = x\eta'' \in L$ and $\eta = x\eta' \in L$. Therefore $\eta = x\eta' = x^2\eta''$, so $x(\eta' - x\eta'') = 0$, and thus

$$0 \neq \eta' - x\eta'' \in L \cap \ker(\cdot x) = L \cap H^{d-1}_{\mathfrak{m}}\left(R/xR\right).$$

• • •

Note $\ker(\cdot x) \subseteq L$. Let $\eta \in \ker(\cdot x)$, so $x^t\eta = 0$ for all $t \gg 0$, and $x^t\eta \in x^tN = L$. As L is F-stable, $\rho^e(x^t\eta) \in L$, so $x^{p^e-1}\rho^e(\eta) \in L$, which is 0, but $x^{p^e-1}\rho^e$ is injective. $\square$

If the claim holds, then $0 \neq L \cap H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \subseteq H^{d-1}_{\mathfrak{m}}\left(R/xR\right)$ is proper and F-stable. Therefore, $H^{d-1}_{\mathfrak{m}}\left(R/xR\right) \subseteq L$.

Next, note that $H^d_{\mathfrak{m}}(R)/L \xrightarrow{\cdot x} H^d_{\mathfrak{m}}(R)/L$ is injective. Indeed, $\ker(\cdot x) = H^{d-1}_{\mathfrak{m}}\left(R/xR\right)$. Write $L = x^t N \neq x^{t-1}N$; then $\eta \in x^{t-1}N \setminus \{0\}$ is in $\ker(\cdot x)$ on $H^d_{\mathfrak{m}}(R)/L$, a contradiction.

If $x\eta \in L$, then $x^{p^e-1}\rho^e(\eta) = L$, but $\ker(\cdot x) \subseteq L$, so the restriction of $x^{p^e-1}\rho^e$ to $H^d_{\mathfrak{m}}(R)/L$ is injective, so $\eta \in L$. $\square$

Remark 1.8.19. F-nilpotent does not deform in general. Let $R = k[x,y,z]_{\mathfrak{m}}/(x^2+y^3+z^7+xyz)$. One can check $R/zR \cong k[x,y]_{\mathfrak{m}}/(x^2+y^3)$ is F-nilpotent (in fact, F-rational), but R is not F-nilpotent.

Remark 1.8.20. What about gluing? Note that F^e_*- is exact as a functor. That is, $H^i_{\mathfrak{m}}(F^e_*R) \cong F^e_*H^i_{\mathfrak{m}}(R)$ as F^e_*R-modules for all i (a fact we have been implicitly using). If R is regular, then

$$F^e_*H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}(R) \otimes_R F^e_*R$$

by flatness of F^e_*R (**Theorem 1.1.24 [Kunz]**). But in general,

$$F^e_*H^{\dim R}_{\mathfrak{m}}(R) \cong H^{\dim R}_{\mathfrak{m}}(R) \otimes_R F^e_*R$$

for any local ring $(R,\mathfrak{m})$, since tensor product is right exact.

Definition 1.8.21 (0^*_M)**.** Let R be a local domain. For an $R\{F\}$-module (M,ρ), define

$$0^*_M = \{m \in M \mid \text{ there exists } c \neq 0 \text{ such that } c\rho^e(m) = 0 \text{ for some } e \gg 0\}.$$

Remark 1.8.22. Recall that $\rho^e : M \to M$ defines an R-linear map $M \to M \otimes F^e_*R$ by $m \mapsto \rho^e(m) \otimes F^e_*1$. For any $c \neq 0$, we have a composition

$$\mu^e_c : M \xrightarrow{\rho^e \otimes F^e_* 1} M \otimes F^e_*R \xrightarrow{\mathrm{id} \otimes \cdot F^e_* c} M \otimes F^e_*R,$$

and $m \in 0^*_M$ if and only if $m \in \ker \mu^e_c$ for $e \gg 0$.

Remark 1.8.23. Given an $R\{F\}$-module (M,ρ), 0^*_M is nilpotent if and only if for each $m \in 0^*_M$, there exists $e \gg 0$ such that $\rho^e(m) = 0$.

Lemma 1.8.24. *For $R\{F\}$-modules A and B, $0^*_{A\oplus B} \cong 0^8_A \oplus 0^*_B$.*

Proof. For $c \neq 0$,

$$\begin{array}{ccccc} A \oplus B & \longrightarrow & (A\oplus B)\otimes F^e_*R & \longrightarrow & (A\oplus B)\otimes F^e_*R \\ & & \wr\| & & \wr\| \\ & & A\otimes F^e_*R \oplus B \otimes F^e_*R & & A\otimes F^e_*R \oplus B \otimes F^e_*R \end{array}$$

$\square$

Theorem 1.8.25 (Maddox-Miller)**.** *If $(R,\mathfrak{m})$ is a domain with ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that*

1. $\dim\left(R/(\mathfrak{a}_1\cap\mathfrak{a}_2)\right) = \dim\left(R/\mathfrak{a}_1\right) = \dim\left(R/\mathfrak{a}_2\right) = \dim\left(R/\mathfrak{a}_1+\mathfrak{a}_2\right)$*, and*
2. $R/\mathfrak{a}_1$*,* $R/\mathfrak{a}_2$*, and* $R/\mathfrak{a}_1+\mathfrak{a}_2$ *are F-nilpotent,*

then $R/\mathfrak{a}_1\cap\mathfrak{a}_2$ is F-nilpotent.

Proof. First, fix $i < \dim R$. We have a short exact sequence

$$0 \to R/\mathfrak{a}_1\cap\mathfrak{a}_2 \to R/\mathfrak{a}_1 \oplus R/\mathfrak{a}_2 \to R/\mathfrak{a}_1+\mathfrak{a}_2 \to 0.$$

Apply $\mathbf{R}\Gamma_{\mathfrak{m}}$ to get the long exact sequence

$$\cdots \to H^{i-1}_{\mathfrak{m}}\left(R/\mathfrak{a}_1+\mathfrak{a}_2\right) \xrightarrow{\delta} H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\cap\mathfrak{a}_2\right) \xrightarrow{\alpha} H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_1\right) \oplus H^i_{\mathfrak{m}}\left(R/\mathfrak{a}_2\right) \to \cdots$$

By splitting up the long exact sequence, we have

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\delta \arrow[r] \arrow[d, phantom, "\cong" description, sloped] & H^i_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \arrow[r] & \operatorname{im}\alpha \arrow[r] & 0 \\
 & \ker\alpha & & &
\end{tikzcd}
$$

Note that for any short exact sequence of $R\{F\}$-modules

$$0 \to M \to N \to P \to 0,$$

if M and P are nilpotent, then N is nilpotent. Since $\operatorname{im}\delta$ and $\operatorname{im}\alpha$ are nilpotent, $H^i_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right)$ is nilpotent. Thus, the case $i < \dim R$ is shown.

Now, assume $i = \dim R = d$. We have

$$H^{d-1}_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right) \xrightarrow{\delta} H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \xrightarrow{\alpha} H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_2}\right).$$

Let $\xi \in 0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_1\cap\mathfrak{a}_2}\right)}$. That is, there exists $c \neq 0$ and $e \gg 0$ such that $c\rho^e(\xi) = 0$. Note

$$\alpha(\xi) \in 0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_2}\right)},$$

as $\alpha(c\rho^e(\xi)) = c\rho^e(\alpha(\xi)) = 0$ using the commutativity of

$$
\begin{tikzcd}
H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \arrow[r, "\alpha"] \arrow[d, "c\rho^e"] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_2}\right) \arrow[d, "c\rho^e"] \\
H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \arrow[r, "\alpha"] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_2}\right)
\end{tikzcd}
$$

By **Lemma 1.8.24**,

$$0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_2}\right)} \cong 0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_1}\right)} \oplus 0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_2}\right)}.$$

One can check that, since $0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_1}\right)}$ and $0^*_{H^d_{\mathfrak{m}}\left({}^{R}\!/_{\mathfrak{a}_2}\right)}$ are nilpotent, there exists $e \gg 0$ such that $\rho^e(\alpha(\xi)) = 0$.

Next, consider

$$
\begin{tikzcd}
 & \xi \arrow[r, mapsto] \arrow[dddd, mapsto] & \alpha(\xi) \arrow[dddd, mapsto] \\[-1em]
H^{d-1}_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right) \arrow[r, "\delta"] \arrow[d, "\rho^e"] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \arrow[r, "\alpha"] \arrow[d, "\rho^e"] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_2}\right) \arrow[d, "\rho^e"] \\
H^{d-1}_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right) \arrow[r] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}\right) \arrow[r] & H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1}\right) \oplus H^d_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_2}\right) \\[-1em]
 & \rho^e(\xi) \arrow[r, mapsto] & 0
\end{tikzcd}
$$

Let $\zeta \in H^{d-1}_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right)$ with $\delta(\zeta) = \xi$. Since $d = \dim\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right)$, $H^{d-1}_{\mathfrak{m}}\left({}^{R}\!\big/_{\mathfrak{a}_1 + \mathfrak{a}_2}\right)$ is nilpotent. Thus, there exists $e' \gg 0$ such that $\rho^{e'}(\zeta) = 0$, and thus

$$\rho^{e+e'}(\xi) = \rho^{e'}\rho^e(\xi) = \rho^{e'}\delta(\zeta) = \delta\rho^{e'}(\zeta) = 0.$$

Thus, ${}^{R}\!\big/_{\mathfrak{a}_1 \cap \mathfrak{a}_2}$ is F-nilpotent, as desired. $\square$

Corollary 1.8.26. *Let X be an equidimensional scheme which is a union of two schemes $X = Y_1 \cup Y_2$. If Y_1, Y_2, and $Y_1 \cap Y_2$ are F-nilpotent, and if $\dim X = \dim Y_1 = \dim Y_2 = \dim Y_1 \cap Y_2$, then X is F-nilpotent.*

Corollary 1.8.27. *The same theorem holds for F-rational.*

Proof. F-rational singularities are Cohen-Macaulay, F-injective, and F-nilpotent. $\square$

1.8.1 Local Algebra

Remark 1.8.28. One may ask: what does controlling singularities buy us? That is, what does knowing deformation, gluing, or quotient results allow us to do?

Remark 1.8.29. Recall that for a local ring $(R, \mathfrak{m}, k)$, each $R/\mathfrak{m}^n$ is a finite dimensional k-vector space.

Definition 1.8.30 (length)**.** Let $(R, \mathfrak{m})$ be a local ring. Define the **length** of $R/\mathfrak{m}^n$, $\lambda\left(R/\mathfrak{m}^n\right)$, to be the k-dimension of $R/\mathfrak{m}^n$. That is,

$$\lambda\left(R/\mathfrak{m}^n\right) = \dim_k \left(R/\mathfrak{m}^n\right).$$

Remark 1.8.31. The function $n \mapsto \lambda\left(R/\mathfrak{m}^n\right)$ is eventually a polynomial in n of degree n^d, where $d = \dim R$. It is called the Hilbert polynomial.

Definition 1.8.32 (multiplicity)**.** One can set

$$e(R) = \lim_{n\to\infty} \frac{d!\lambda\left(R/\mathfrak{m}^n\right)}{n^d}.$$

Call $e(R)$ the (Hilbert-Samuel) **multiplicity** of $(R, \mathfrak{m})$.

Example 1.8.33. Let $R = k[x_1, ..., x_d]_\mathfrak{m}$. We have $\lambda\left(R/\mathfrak{m}^n\right) = \binom{n+d}{d} = \frac{n^d}{d!} + O(n^{d-1})$, so $e(R) = 1$.

Example 1.8.34. If $R = k[x, y]_\mathfrak{m}/(y^2 - x^2 - x^3)$, then $e(R) = 2$.

Example 1.8.35. If $R = k[x, y]_\mathfrak{m}/(y^2 - x^3)$, then $e(R) = 2$.

Example 1.8.36. If $R = k[x, y]_\mathfrak{m}/(y^{31} - x^{10})$, then $e(R) = 10$.

Remark 1.8.37. A larger multiplicity $e(R)$ implies a worse singularity of R.

Remark 1.8.38. One can show that $\lambda\left(R/\mathfrak{m}^n\right) = \lambda\left(\widehat{R}/\mathfrak{m}^n\widehat{R}\right)$. Thus, by **Theorem 1.3.7 [Cohen Structure Theorem]**, if R is regular, then $e(R) = 1$. The converse does not hold.

Example 1.8.39. If $R = k[x, y, z]_\mathfrak{m}/(xy, xz)$, then $e(R) = 1$, but R is not regular.

Theorem 1.8.40 (Nagata)**.** *Let $\widehat{R}$ be equidimensional. R is regular if and only if $e(R) = 1$.*

Theorem 1.8.41 (Huneke-Watanabe)**.** *Let $(R, \mathfrak{m})$ be a local ring of dimension d and embedding dimension ν.*

1. *If R is F-split, then $e(R) \leq \binom{\nu}{d}$.*
2. *If R is F-rational, then $e(R) \leq \binom{\nu-1}{d-1}$.*

Definition 1.8.42 (reduction)**.** A **reduction** of $\mathfrak{m}$ is an ideal J such that $\mathfrak{m}^n = J\mathfrak{m}^{n-1}$ for $n \gg 0$.

Definition 1.8.43 (minimal reduction)**.** We call a reduction J **minimal** if it is minimal with respect to inclusion. That is, if J' is any other reduction of $\mathfrak{m}$, then $J \subseteq J'$.

Remark 1.8.44 (Brianǫn-Skoda)**.** A theorem by Brianǫn-Skoda has the following consequence: if $(R, \mathfrak{m})$ is F-rational of dimension d and J is a minimal reduction of $\mathfrak{m}$, then $\mathfrak{m}^d \subseteq J$. Additionally, one can fairly easily show that if $(R, \mathfrak{m})$ is F-split, then $\mathfrak{m}^{d+1} \subseteq J$.

Remark 1.8.45. The above tools are used to prove **Theorem 1.8.41**. Set minimal generators $x_1, ..., x_d, y_1, ..., y_{\nu-d}$ for $\mathfrak{m}$. Let $J = (x_1, ..., x_d)$ be a minimal reduction. Consequently, if $(R, \mathfrak{m})$ is F-rational, then ${}^{R}\!/_{J}$ has k-span comprised of monomials in $y_1, ..., y_{\nu-d}$ of degree at most $d-1$. If $(R, \mathfrak{m})$ is F-split, then ${}^{R}\!/_{J}$ has k-span comprised of monomials in $y_1, ..., y_{\nu-d}$ of degree at most d. This gives

$$\lambda\left({}^{R}\!/_{J}\right) \leq \begin{cases} \binom{\nu-1}{d-1} & \text{if } (R, \mathfrak{m}) \text{ is } F\text{-rational;} \\ \binom{\nu}{d} & \text{if } (R, \mathfrak{m}) \text{ is } F\text{-split.} \end{cases}$$

Theorem 1.8.46 (Katzman-Zhang)**.** *If $(R, \mathfrak{m})$ is a Cohen-Macaulay F-injective local ring, then $e(R) \leq p^\eta \binom{\nu}{d}$, for a specific η.*

Remark 1.8.47. The proof of **Theorem 1.8.46** uses an important fact. For an $R\{F\}$-module (M, ρ), set

$$0_M^\rho = \{m \in M \mid \rho^e(m) = 0 \text{ for some } e\} \subseteq M.$$

Set

$$HSL(M) = \inf_e \{0_M^\rho = \ker \rho^e\},$$

which need not be finite. (Indeed, consider $M = H_{\mathfrak{a}}^i(R)$. If M is not noetherian or artinian, $\{\ker \rho^e\}$ may fail to stabilize.)

Theorem 1.8.48 (Hartshorne-Speiser-Lyubeznik)**.** *If (M, ρ) is artinian, then $HSL(M) < \infty$.*

Remark 1.8.49. The specific η in **Theorem 1.8.46** is $\eta = \max_i \{HSL(H_{\mathfrak{m}}^i(R))\}$.

Remark 1.8.50. A natural question is the following: if $(R, \mathfrak{m})$ is F-nilpotent, is $e(R)$ bounded by some function of d, ν, and η?

1.9 F-regular Rings

Remark 1.9.1. There is another way to characterize $0^*_{H_{\mathfrak{m}}^d(R)} = 0$. Let R be a Cohen-Macaulay domain. For each $c \neq 0$, if $c\rho^e$ is injective for some $e \gg 0$, then by construction, $0^*_{H_{\mathfrak{m}}^d(R)} = 0$.

Definition 1.9.2 (strongly F-regular)**.** A domain $(R, \mathfrak{m})$ is **strongly F-regular** provided that for each $c \neq 0$, there exists $e \gg 0$ such that

$$\begin{aligned} R &\to F_*^e R \\ 1 &\mapsto F_*^e c \end{aligned}$$

splits. (Slogan: a strongly F-regular ring has lots of splittings.)

Remark 1.9.3. For now, we drop "strongly," and refer to such rings as F-regular. Be warned that this will eventually clash with **Definition 1.13.106 [F-regular]**.

Example 1.9.4. Regular rings are F-regular.

Example 1.9.5. Though we do not yet have the tools to verify this, the ring $R = {}^{k[x,y,z]_{\mathfrak{m}}}\!/_{(x^2+y^2+z^2)}$ is not regular, though it is F-regular.

Theorem 1.9.6. *If $(R, \mathfrak{m})$ is F-regular, then it is F-rational.*

Proof. We first show that $(R, \mathfrak{m})$ is Cohen-Macaulay. To see this, we will use a fact to be proven later (**Corollary 1.10.48**), using local/Matlis duality:

Claim. *For each $i < \dim R$, there exists $c \neq 0$ such that $cH^i_{\mathfrak{m}}(R) = 0$; i.e.,* $\operatorname{Ann} H^i_{\mathfrak{m}}(R) \neq 0$.

Assuming this claim, choose $e > 0$ such that

$$\begin{array}{ccccc} R & \longrightarrow & F^e_* R & \xrightarrow{\cdot F^e_* c} & F^e_* R \\ 1 & \longmapsto & F^e_* 1 & \longmapsto & F^e_* c \end{array}$$

splits. Apply $H^i_{\mathfrak{m}} -$ to get

$$H^i_{\mathfrak{m}}(R) \to F^e_*(cH^i_{\mathfrak{m}}(R)) = 0,$$

which is injective, so $H^i_{\mathfrak{m}}(R) = 0$. For each $c \neq 0$, $c\rho^e$ is injective on $H^{\dim R}_{\mathfrak{m}}(R)$, so we have $0^*_{H^{\dim R}_{\mathfrak{m}}(R)} = 0$, so $(R, \mathfrak{m})$ is F-rational. $\square$

Remark 1.9.7. It's easy to see that if $R \hookrightarrow S$ is a split extension of domains with S an F-regular ring, then R is F-regular. It's also clear that F-regular implies F-split, by choosing $c = 1$.

Remark 1.9.8. Once more, we may add to the diagram from **Remark 1.8.16**:

$$\begin{tikzcd}
F\text{-regular} \arrow[rr, Rightarrow, "\textbf{Theorem 1.9.6}"] \arrow[d, Rightarrow, "\textbf{Remark 1.9.7}"'] & & F\text{-rational} \arrow[d, Rightarrow, bend right, "\textbf{Theorem 1.8.4}"'] \\
F\text{-split} \arrow[rr, Rightarrow, "\textbf{Lemma 1.7.1}"] \arrow[dr, Rightarrow, "\textit{Proof of } \textbf{\textit{Theorem 1.6.3}} \textit{ [Ma]}"'] & & F\text{-injective} \arrow[u, Rightarrow, bend right, "+F\text{-nilpotent, } \textbf{Theorem 1.8.15 [Srinivas-Takagi]}"'] \\
& \text{anti-nilpotent} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Remark 1.9.9. The implication in **Remark 1.9.7** cannot be reversed. The ring $R = {}^{k[x,y,z]}\!/\!_{(x^3+y^3+z^3)}$ is F-split for $p \equiv 1 \bmod 3$, but R is not F-regular.

Theorem 1.9.10 (Glassbrenner's Criterion)**.** *If $(S, \mathfrak{m})$ is a regular and $\mathfrak{p} \subseteq S$ is a prime ideal, then $R = {}^{S}\!/\!_{\mathfrak{p}}$ is F-regular if and only if for each $c \notin \mathfrak{p}$, there exists $e > 0$ such that $c(\mathfrak{p}^{[p^e]} : \mathfrak{p}) \not\subseteq \mathfrak{m}^{[p^e]}$.*

Remark 1.9.11. Notice the similarities to **Corollary 1.4.24 [Fedder's Criterion]**. The proof is similar.

Theorem 1.9.12. *Being F-regular is a local property. That is, for a domain R, R is F-regular if and only if $(R_{\mathfrak{m}}, \mathfrak{m}, k)$ is F-regular for all maximal ideals $\mathfrak{m}$.*

Proof. It's clear that R is F-regular implies $(R_{\mathfrak{m}}, \mathfrak{m}, k)$ is F-regular for all $\mathfrak{m}$.

On the other hand, set $c \neq 0$. Fix a maximal ideal $\mathfrak{m}$. The map $R_{\mathfrak{m}} \to F^e_* R_{\mathfrak{m}}$ defined by $1 \mapsto F^e_* c$ splits for $e \gg 0$, and the value of e depends on $\mathfrak{m}$. That is,

$$\operatorname{Hom}_{R_{\mathfrak{m}}}(F^e_* R_{\mathfrak{m}}, R_{\mathfrak{m}}) \xrightarrow{ev_{F^e_* c}} R_{\mathfrak{m}}$$

is surjective for $e \gg 0$ depending on $\mathfrak{m}$. Pick a neighborhood $U_{\mathfrak{m}} \subseteq \max\operatorname{Spec} R$ so that $ev_{F^e_* c}$ is surjective for all $\mathfrak{n} \in U_{\mathfrak{m}}$; i.e., there is one value of e that works for all $\mathfrak{n} \in U_{\mathfrak{m}}$.

Note the following topological fact: $\max\operatorname{Spec} R$ is quasi-compact in the Zariski topology.

Pick a finite subcover of $\{U_{\mathfrak{m}}\}$ and take a single e that works to have $ev_{F^e_* c}$ surjective for all $\mathfrak{m}$. $\square$

Remark 1.9.13. Note that in **Theorem 1.4.55 [Grifo-Huneke]**, we saw that symbolic powers were a useful tool in studying F-split rings. Namely, we saw that if S is a regular ring, $\mathfrak{a} \subseteq S$ is an ideal with $\operatorname{bight} \mathfrak{a} = h$, and ${}^{S}\!/\!_{\mathfrak{a}}$ is F-split, then $\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n$ for $n \geq 1$. The proof used **Corollary 1.4.24 [Fedder's Criterion]**. One might wonder: can we use **Theorem 1.9.10 [Glassbrenner's Criterion]** to show something similar for F-regular rings?

Theorem 1.9.14 (Grifo-Huneke)**.** *If S is a regular ring, $\mathfrak{p} \subseteq S$ is a prime ideal with $\operatorname{ht} \mathfrak{p} = h \geq 2$, and ${}^{S}\!/\!_{\mathfrak{p}}$ is F-regular, then for $n \geq 1$, $\mathfrak{p}^{(n(h-1)+1)} \subseteq \mathfrak{p}^{n+1}$.*

Corollary 1.9.15. *If $h = 2$, then $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for all $n \geq 2$.*

Theorem 1.9.16. *Let S be F-regular. If $S \subseteq R$ is a module finite extension; i.e., R is a finitely generated S-module, then the extension splits.*

Definition 1.9.17 (splinters)**.** A ring S is called a **splinter** provided if $S \subseteq R$ is a module finite extension, then the extension splits.

Remark 1.9.18. Hochster stated the Direct Summand Conjecture: every regular ring in any setting (characteristic $p > 0$, characteristic 0, mixed characteristic) is a splinter. Some specific cases are quite tractable. If $S = k[x_1, ..., x_n]$ with $\operatorname{char} k = 0$, then S is a splinter. For any extension $S \subseteq R$ of domains, there is a trace map on fraction fields:

$$\begin{gathered}\operatorname{Frac} R \xrightarrow{Tr} \operatorname{Frac} S\\ x \mapsto Trx.\end{gathered}$$

It's not hard to see $\operatorname{im} Tr|_R \subseteq S$. Set $n = [\operatorname{Frac} R : \operatorname{Frac} S]$. Consequently,

$$\begin{array}{ccccc} S & \longrightarrow & R & \xrightarrow{\frac{1}{n}Tr} & S \\ 1 & \longmapsto & 1 & \longmapsto & \frac{1}{n}n \end{array}$$

is a splitting. Hochster additionally proved the Direct Summand Conjecture in characteristic p. The difficult case was mixed characteristic.

Definition 1.9.19 (mixed characteristic)**.** A local ring $(R, \mathfrak{m}, k)$ is **mixed characteristic** if $\operatorname{char} R = 0$ but $\operatorname{char} k = p > 0$.

Example 1.9.20. $(\mathbf{Z}_{(p)}, (p), \mathbf{F}_p)$ is mixed characteristic. The p-adics are also mixed characteristic, as are polynomials over these, quotients, etc.

Theorem 1.9.21 (André)**.** *The Direct Summand Conjecture holds.*

Remark 1.9.22. The proof uses Scholze's perfectoid spaces, coming from number theory and p-derivations. The methods are similar, in some sense, to the proof of **Theorem 1.1.24 [Kunz]**, using $R^{perf} = \varinjlim_F R$.

Remark 1.9.23. Note that all regular domains are F-regular in characteristic $p > 0$. To see this, for any $c \neq 0$, pick a basis for $F^e_* R$ where $F^e_* c$ is a basis element. Hence we can prove the Direct Summand Conjecture in positive characteristic by showing the following.

Theorem 1.9.24. *F-regular rings are splinters.*

Proof. Suppose S is F-regular, and suppose $S \subseteq R$ is module finite. For simplicity, assume R is a domain, and identify the Frobenius as $R \to R^{\frac{1}{p^e}}$ and $S \to S^{\frac{1}{p^e}}$ via $p^{e^{th}}$ roots. For $e > 0$ and a map $\varphi : S^{\frac{1}{p^e}} \to S$, we get a diagram

$$\begin{array}{ccc} \psi & \longmapsto & R \hookrightarrow R^{\frac{1}{p^e}} \xrightarrow{\psi} S^{\frac{1}{p^e}} \xrightarrow{\varphi} S \\ \operatorname{Hom}\left(R^{\frac{1}{p^e}}, S^{\frac{1}{p^e}}\right) & \longrightarrow & \operatorname{Hom}(R, S) \\ \Big\downarrow {\scriptstyle ev_{1^{\frac{1}{p^e}}}} & & \Big\downarrow {\scriptstyle ev_1} \\ S^{\frac{1}{p^e}} & \xrightarrow[\varphi]{} & S \end{array}$$

Our goal is to show that ev_1 is surjective. Pick $c \neq 0$ in $\operatorname{im} ev_1$, which exists as R and S are domains. We can consider

$$\operatorname{Hom}(\operatorname{Frac} R, \operatorname{Frac} S) \xrightarrow{\operatorname{Frac} ev_1} \operatorname{Frac} S,$$

which is surjective because $\operatorname{Frac} R \subseteq \operatorname{Frac} S$ splits, as they're fields. Pick a nonzero $c \in \operatorname{im}(\operatorname{Frac} ev_1)$ and clear denominators. Pick $e \gg 0$ and φ so that $\varphi\left(c^{\frac{1}{p^e}}\right) = 1$. This makes the composition

$$\varphi ev_{1^{\frac{1}{p^e}}} : \operatorname{Hom}\left(R^{\frac{1}{p^e}}, S^{\frac{1}{p^e}}\right) \to S$$

a surjection, and thus by the diagram above, $\operatorname{Hom}\left(R^{\frac{1}{p^e}}, S^{\frac{1}{p^e}}\right) \to \operatorname{Hom}(R, S) \xrightarrow{ev_1} S$ is a surjection, and therefore ev_1 is surjective, as desired. $\square$

Remark 1.9.25. One might ask if the converse to **Theorem 1.9.24** holds. Are all splinters in characteristic $p > 0$ F-regular rings? This is an open problem.

1.10 Local Duality and Gorenstein Rings

Remark 1.10.1. Our goal now is to take the diagram from **Remark 1.9.8** and begin to add hypotheses that will reverse some of the other implications.

Remark 1.10.2. Recall **Definition 1.5.43** [**essential submodule** of an essential extension of R-modules $M \subseteq E$. We have seen in **Remark 1.5.45** and **Remark 1.5.5** that $\operatorname{soc} H^i_{\mathfrak{m}}(R) \subseteq H^i_{\mathfrak{m}}(R)$ is an essential extension.

Definition 1.10.3 (injective module)**.** An injective module is a module E such that $\operatorname{Hom}(-, E)$ is exact. Equivalently, given any injection $N \to M$ and map $N \to E$, there exists a map $M \to E$ making the following diagram commute.

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & N \arrow[u] \arrow[r] & M \arrow[ul, dashed] \end{tikzcd}$$

Remark 1.10.4. The category R-**mod** has enough injectives; i.e., every R-module M embeds in an injective module E.

Definition 1.10.5 (injective hull)**.** Every module M has an essential extension $M \subseteq E$ with E an injective module. Such an E is unique up to unique isomorphism. Denote this isomorphism class by $E(M)$ and call $E(M)$ the **injective hull** of M.

Remark 1.10.6. Let $(R, \mathfrak{m}, k)$ be a local ring. Special attention is paid to $E(k)$, the injective hull of the residue field. We write $E_R(k)$ for $E(k)$.

Example 1.10.7.
1. $E(\mathbf{Z}) = \mathbf{Q}$.
2. $E\left(\mathbf{Z}/p\mathbf{Z}\right) = \mathbf{Z}[p^{-1}]/\mathbf{Z}$.

Example 1.10.8. If $(R, \mathfrak{m}, k)$ is a 1-dimensional domain with $\operatorname{Frac} R = K$, then there is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & K \arrow[r] & E_R(k) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\ & & & K/R & \end{tikzcd}$$

Example 1.10.9. Let $R = k[x]_{\mathfrak{m}}$, one can write $E_R(k)$ as $E_R(k) \cong x^{-1}k[x^{-1}] \cong H^1_{\mathfrak{m}}(R)$. A similar statement holds for $R = k[x_1, ..., x_d]_{\mathfrak{m}}$; i.e., $E_R(k) \cong H^d_{\mathfrak{m}}(R)$.

Remark 1.10.10. We have the following basic facts about injective modules:
- Any direct sum of injective modules is injective.
- Any direct summand of an injective module is injective.

Lemma 1.10.11. *Let E be an R-module. The following are equivalent:*

1. *E is injective,*
2. *every injection $E \to N$ splits, and*
3. *E has no proper essential extensions.*

Proof. For 1 implies 2, let $E \to N$ be an injection and use the lift given by

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "\mathrm{id}"] \arrow[r] & N \arrow[ul, dashed] \end{tikzcd}$$

Hence $N \to E$ is a splitting.

For 2 implies 3, if $E \subseteq N$, then $N \cong E \oplus E'$. This is essential if and only if $E' = 0$, by **Definition 1.5.43 [essential submodule]**.

For 3 implies 1, assume for the sake of contradiction that E is not injective. Pick $E \subseteq E'$ with E' injective. This is not an essential extension, so “Zornify” the set

$$\{M \subseteq E' \mid M \cap E = 0\}$$

to get a maximal $N \subseteq E'$ such that $E \cap N = 0$. Thus $E \to E'/N$ is essential, and $E' \cong E \oplus N$. Thus, E is injective, a contradiction. ☐

Lemma 1.10.12. *If $(R, \mathfrak{m}, k)$ is a local ring, then $E_R(k)$ is $\mathfrak{m}$-torsion and $\operatorname{Hom}_R(k, E_R(k)) \cong k$.*

Proof. It is clear that $\operatorname{Ass}(k) = \operatorname{Ass}(E_R(k))$, as any $x \in E$ with $E/xE \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$ has a multiple in $k \subseteq E_R(k)$. Hence $R/\mathfrak{p} \cong R/\mathfrak{m}$. Thus $E_R(k)$ is $\mathfrak{m}$-torsion, as desired.

Next, note that $k \subseteq (0 :_E \mathfrak{m}) \subseteq E_R(k)$, but if $k \neq (0 :_E \mathfrak{m})$, then the first inclusion splits, yet this would contradict the fact that $E_R(k)$ is essential via **Lemma 1.10.11**. Thus, $\operatorname{Hom}_R(k, E_R(k)) \cong (0 :_E \mathfrak{m}) = k$. ☐

Remark 1.10.13. One application of the fact that R-**mod** has enough injectives, and has injective hulls in particular, is that we can use them to build injective resolutions. The resolution

$$\begin{tikzcd}[column sep=small] & & & 0 \arrow[dr] & & 0 & & & \\ & & & & K_0 \arrow[ur] \arrow[dr] & & & & \\ 0 \arrow[r] & M \arrow[r] & E^0 = E(M) \arrow[rr] \arrow[urr] & & & E^1 = E(K^0) \arrow[rr] \arrow[dr] & & E^2 = E(K^1) \arrow[r] & \cdots \\ & & & & & & K_1 \arrow[ur] \arrow[dr] & & \\ & & & & & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

is *the* minimal injective resolution of M.

Theorem 1.10.14 (Bass Structure Theorem)**.** *If R is noetherian and E is injective, then*

$$E \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R\left(R/\mathfrak{p}\right)^{\oplus \mu_{\mathfrak{p}}}$$

where

$$\mu_{\mathfrak{p}} = \dim_{R/\mathfrak{p}} \operatorname{Hom}_{R_{\mathfrak{p}}}\left(R/\mathfrak{p}, E_R\left(R/\mathfrak{p}\right)\right).$$

Additionally, for the minimal injective resolution $0 \to M \to E^\bullet$, one has

$$E^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R\left({}^{R}\!/\!_{\mathfrak{p}} \right)^{\oplus \mu(i,\mathfrak{p})}.$$

One calls the number $\mu(i,\mathfrak{p})$ the Bass number. For a module M, we have

$$\mu(i,\mathfrak{p})(M) = \dim \operatorname{Ext}^i(k(\mathfrak{p}), M_{\mathfrak{p}}).$$

Example 1.10.15. For $R = {}^{S}\!/\!_{\mathfrak{a}}$ with $S_{\mathfrak{m}}$ a localization of a polynomial ring of dimension n, one has

$$\lambda_{ij} = \mu(i,\mathfrak{m})(H^{n-j}_{\mathfrak{a}}(R)) = \dim \operatorname{Ext}^i_S\left(k, H^{n-j}_{\mathfrak{a}}(R)\right).$$

The numbers λ_{ij} are called Lyubeznik numbers. These are independent of the presentation of R, and they capture topological information about $\operatorname{Spec} R$.

Remark 1.10.16. A pressing question: how can we actually calculate $E_R(k)$ for a given local ring $(R,\mathfrak{m},k)$?

Theorem 1.10.17. *If $(S,\mathfrak{m},k) \to (R,\mathfrak{n},\ell)$ is a map of local rings, then* $\operatorname{Hom}_S(R, E_S(k)) \cong E_R(\ell)$.

Proof. First suppose $R \cong {}^{S}\!/\!_{\mathfrak{a}}$. If E is an injective S-module, then $\operatorname{Hom}_R(R,E)$ is an injective R-module. This is because R is flat, so $- \otimes_S R$ is exact, and by hom-tensor adjunction, $\operatorname{Hom}_S(-,E)$ is exact.

Also using hom-tensor adjunction, we have

$$\operatorname{Hom}_R(-, \operatorname{Hom}_S(R,E)) \cong \operatorname{Hom}_R(-,E).$$

Now consider the case that $(S,\mathfrak{m},k) \twoheadrightarrow (R,\mathfrak{n},\ell)$ with $\mathfrak{m}R = \mathfrak{n}$ (i.e., a local extension). In other words, assume S is a localization of a polynomial ring. Apply the above calculation to $E = E_S(k)$, and see that $\operatorname{Hom}_R(R, E_S(k))$ is an injective R-module. It is also clearly $\mathfrak{n}$-torsion, since $E_S(k)$ is $\mathfrak{m}$-torsion.

By **Theorem 1.10.14 [Bass Structure Theorem]**, $\mu_{\mathfrak{p}}(\operatorname{Hom}_S(R, E_S(k)) = 0$ unless $\mathfrak{p} = \mathfrak{n}$. Therefore,

$$\operatorname{Hom}_S(R, E_S(k)) \cong E_R(\ell)^{\oplus t}.$$

The result follows if we can show $t = 1$.

Note that $t = \dim \operatorname{Hom}_R(\ell, \operatorname{Hom}_S(R, E_S(k)))$ and

$$\begin{aligned}
\operatorname{Hom}_R(\ell, \operatorname{Hom}_S(R, E_S(k))) &\cong \operatorname{Hom}_S(\ell \otimes_R R, E_S(k)) \\
&\cong \operatorname{Hom}_S(\ell \otimes k, E_S(k)) \\
&\cong \operatorname{Hom}_k(\ell \otimes k, E_S(k)) \\
&\cong \operatorname{Hom}_k(\ell, \operatorname{Hom}(k, E_S(k))) \\
&\cong \operatorname{Hom}_k(\ell, k).
\end{aligned}$$

Thus, $t = 1$, as desired. □

Remark 1.10.18. Recall that for a finite dimensional k-vector space V, there is a canonical isomorphism $V^\vee = \operatorname{Hom}_k(V,k) \cong V$. Furthermore, $V \cong (V^\vee)^\vee$. One might generalize from vector spaces over a field to modules over a ring, and ask if, given an R-module M, is $\operatorname{Hom}_R(M,R) \cong M$?

Theorem 1.10.19 (Matlis)**.** *Let $(R,\mathfrak{m},k)$ be a local ring. Set $(-)^\vee = \operatorname{Hom}_R(-,E)$ where $E = E_R(k)$.*

1. *The functor $(-)^\vee$ is contravariant and fully faithful.*
2. *If N is artinian, then $N^\vee$ is noetherian, and $N^{\vee\vee} \cong N$.*
3. *If N is noetherian, then $N^\vee$ is artinian, and $N^{\vee\vee} \cong \widehat{N}$.*
4. *When R is complete, $(-)^\vee$ induces an equivalence of categories*

$$\{\textit{noetherian } R\textit{-modules}\} \xleftrightarrow{\sim} \{\textit{artinian } R\textit{-modules}\}.$$

Remark 1.10.20. Since E is injective, $(-)^\vee$ is exact.

Remark 1.10.21. Matlis duality, along with the following fancier (i.e., derived) duality, will be powerful tools.

Definition 1.10.22 (dualizing complex)**.** For a noetherian ring R, a **dualizing complex** is an object $\omega_R^\bullet \in \operatorname{obj}(D(R))$ such that

1. $\omega_R^\bullet$ is quasi-isomorphic to a bounded complex of injective modules, and
2. the natural map $C^\bullet \to \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(C^\bullet, \omega_R^\bullet), \omega_R^\bullet)$ is a quasi-isomorphism for all $C^\bullet \in \operatorname{obj}(D(R))$.

Remark 1.10.23. Condition 2 above is notably hard to check, but it can be replaced. It is equivalent to say that $\omega_R^\bullet$ is a dualizing complex provided 1 holds and that $\mathbf{R}\operatorname{Hom}(\omega_R^\bullet, \omega_R^\bullet) \cong R$. Roughly, to see this, use

$$\mathbf{R}\operatorname{Hom}_R(\omega_R^\bullet, \omega_R^\bullet) \otimes^{\mathbf{L}} C^\bullet \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(C^\bullet, \omega_R^\bullet), \omega_R^\bullet).$$

This is a fancy derived version of saying that if E and P are modules, then the natural map

$$\begin{gathered}\operatorname{Hom}(E,E) \otimes P \to \operatorname{Hom}(\operatorname{Hom}(P,E),E)\\ \varphi \otimes p \mapsto (\psi \mapsto \varphi(\psi(p)))\end{gathered}$$

is an isomorphism.

Remark 1.10.24. Any shift of a dualizing complex is a dualizing complex. Furthermore, if $\omega_R^\bullet$ is a dualizing complex, then $\omega_R^\bullet \otimes P$ for any rank 1 projective module P is also a dualizing complex.

Definition 1.10.25 (normalized dualizing complex)**.** A dualizing complex $\omega_R^\bullet$ is **normalized** provided $h^{-\dim R}(\omega_R^\bullet) \neq 0$ and $h^{-i}(\omega_R^\bullet) = 0$ for $i > \dim R$.

Definition 1.10.26 (canonical module)**.** Given a normalized dualizing complex of R, $\omega_R^\bullet$, the **canonical module** is $\omega_R = h^{-\dim R}(\omega_R^\bullet) \neq 0$.

Remark 1.10.27. Note that not all rings have dualizing complexes. However, any ring essentially of finite type over a field does.

Example 1.10.28. We compute a dualizing complex for $R = {}^{S}\!/_{\mathfrak{a}}$ when S is regular. First, note that when S is regular, $\omega_S^\bullet = S[\dim S]$ is a normalized dualizing complex. To see this, note that S has a finite injective resolution, as S is regular. Furthermore,

$$h^i(\mathbf{R}\operatorname{Hom}(\omega_S^\bullet, \omega_S^\bullet)) = 0$$

unless $i = -\dim S$, in which case

$$h^{-\dim S}(\mathbf{R}\operatorname{Hom}(\omega_S^\bullet, \omega_S^\bullet)) = \operatorname{Hom}(S,S) \cong S.$$

Recall that

$$h^i(\mathbf{R}\operatorname{Hom}(C^\bullet, D^\bullet)) \cong \operatorname{Ext}^i(C^\bullet, D^\bullet).$$

We claim that $\mathbf{R}\operatorname{Hom}_S(R, \omega_S^\bullet)$ is a dualizing complex for R. Roughly, S is quasi-isomorphic to a bounded complex of injectives, and $\operatorname{Hom}_S(R,E)$ is an injective R-module when E is an injective S-module, by hom-tensor adjunction. That is, $\mathbf{R}\operatorname{Hom}(R, \omega_S^\bullet)$ is quasi-isomorphic to a bounded complex of injectives. Also,

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(R,\omega_S^\bullet), \mathbf{R}\operatorname{Hom}_S(R,\omega_S^\bullet)) &\cong \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(R,\omega_S^\bullet) \otimes_R^{\mathbf{L}} R, \omega_S^\bullet)\\ &\cong \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(R,S) \otimes_R^{\mathbf{L}} R, S)\\ &\cong \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(R,S), S)\end{aligned}$$

As S is a dualizing complex, we get

$$\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(R,S), S) \cong S.$$

Example 1.10.29. We can also compute a canonical module. By flatness of localization, any localization of a dualizing complex is again a dualizing complex. As a consequence, every ring essentially of finite type over a field has a canonical module. Indeed, when $R \cong {}^{S}\!/\!_{\mathfrak{a}}$ with S a polynomial ring,

$$\omega_R = h^{-\dim R}(\omega_R^\bullet) = h^{-\dim R}(\mathbf{R}\operatorname{Hom}(R, S[\dim S])) = h^{\dim S - \dim R}(\mathbf{R}\operatorname{Hom}(R,S)) = \operatorname{Ext}^{\dim S - \dim R}(R,S).$$

Remark 1.10.30. We have not yet dealt with completion in the derived sense. Since M is complete when $M \to \widehat{M}$ is an isomorphism, what does it mean to ask for the completion of $M^\bullet$ a complex? We can make sense of

$$\varprojlim_n {}^{M^\bullet}\!/\!_{\mathfrak{m}^n M^\bullet},$$

but $\varprojlim -$ is not exact. Furthermore, for local rings which are not noetherian, a cokernel of $\widehat{M} \to \widehat{N}$ between complete modules can fail to be complete. In fact, $\bigcap_{n \geq 1} \mathfrak{m}^n$ need not be 0.

Remark 1.10.31. There is a more robust notion of completion, not just $\mathbf{R}\varprojlim$. Recall that for an R-module M,

$${}^{M}\!/\!_{\mathfrak{m}^n M} \cong M \otimes_R {}^{R}\!/\!_{\mathfrak{m}^n R}.$$

We can set $\mathfrak{m} = (f_1, ..., f_s)$ and view M as a $\mathbf{Z}[x_1, ..., x_s]$-module via $\mathbf{Z}[x_1, ..., x_s] \to R$ where $x_i \mapsto f_i$.

Definition 1.10.32 (derived complete)**.** Call M **derived complete** provided that the natural map

$$M \to \widehat{M}^{der} = \mathbf{R}\varprojlim_n \left(M \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,...,x_s]} {}^{\mathbf{Z}[x_1,...,x_s]}\!/\!_{(x_1,...,x_s)^n} \right)$$

is a quasi-isomorphism.

Remark 1.10.33. In the case that $\bigcap_{n \geq 1} \mathfrak{m}^n = 0$, then $\widehat{M} \cong \widehat{M}^{der}$.

Lemma 1.10.34 (Derived Nakayama's Lemma)**.** *Let $\mathfrak{a}$ be any ideal in any ring R. Let M be an $\mathfrak{a}$-derived complete module. If ${}^{M}\!/\!_{\mathfrak{a}M} = 0$, then $M = 0$.*

Remark 1.10.35. Since the map $M \to \widehat{M}^{der}$ is faithfully flat, $\widehat{M}^{der} \cong M \otimes^{\mathbf{L}} \widehat{R}^{der}$.

Theorem 1.10.36 (Local duality)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring with a dualizing complex $\omega_R^\bullet$. Let $E = E_R(k)$. For any complex $C^\bullet$,*

$$\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}(C^\bullet, \omega_R^\bullet), E) \cong \mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet).$$

Remark 1.10.37. Applying (derived) Matlis duality to local duality, we get

$$\mathbf{R}\widehat{\operatorname{Hom}(C^\bullet, \omega_R^\bullet)} \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet), E).$$

When R is complete, this becomes

$$\mathbf{R}\operatorname{Hom}(C^\bullet, \omega_R^\bullet) \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet), E).$$

Remark 1.10.38. Local duallity has a classical statement. If we apply $h^{-i}-$ to the above conclusion, we get

$$h^{-i}\mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet), E) \cong \operatorname{Ext}^{-i}(\mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet), E) \cong \operatorname{Hom}(h^i(\mathbf{R}\Gamma_{\mathfrak{m}}(C^\bullet)), E).$$

Thus, local duality gives

$$\operatorname{Ext}^{-i}(C^\bullet, \omega_R^\bullet) \cong \operatorname{Hom}(H^i_{\mathfrak{m}}(C^\bullet), E).$$

We can apply this to $C^\bullet \cong R[0]$. We get

$$\operatorname{Ext}^{-i}(R, \omega_R^\bullet) \cong \operatorname{Hom}(H^i_\mathfrak{m}(R), E).$$

See that, if R is Cohen-Macaulay, then $H^i_\mathfrak{m}(R) = 0$ for $i < \dim R$, so this forces (in fact, it is equivalent to) $\operatorname{Ext}^{-i}(R, \omega_R^\bullet) = 0$ for $i < \dim R$. This occurs if and only if $\omega_R^i = 0$ for $i \neq \dim R$. That is, R is Cohen-Macaulay if and only if $\omega_R^\bullet \cong_q \omega_R[\dim R]$.

Corollary 1.10.39. *If R is Cohen-Macaulay, $\omega_R^\bullet \cong_q \omega_R[\dim R]$, and for any R-module M,*

$$\operatorname{Ext}^{d-i}(M, \omega_R) \cong \operatorname{Hom}(H^i_\mathfrak{m}(M), E).$$

Remark 1.10.40. We get an even stronger result when $\omega_R \cong R$. For example,

$$\operatorname{Hom}(H^d_\mathfrak{m}(R), R) \cong \operatorname{Ext}^0(R, R) = \operatorname{Hom}(R, R) \cong R.$$

That is, via another Matlis duality, $H^d_\mathfrak{m}(R) \cong \operatorname{Hom}(R, E) \cong E$.

Definition 1.10.41 (quasi-Gorenstein)**.** A local ring $(R, \mathfrak{m})$ with a dualizing complex $\omega_R^\bullet$ is **quasi-Gorenstein** (also **1-Gorenstein**) if $\omega_R \cong R$.

Definition 1.10.42 (Gorenstein)**.** A local ring $(R, \mathfrak{m})$ with a dualizing complex $\omega_R^\bullet$ is **Gorenstein** if R is quasi-Gorenstein and Cohen-Macaulay.

Example 1.10.43. If R is a regular local ring, then R is Gorenstein.

Example 1.10.44. Hypersurfaces (complete intersections) are Gorenstein.

Example 1.10.45. Let $S = {}^{k[x,y,z,a,b,c]}\!/\!_{(x^3, a^3)}$. Let R be the subalgebra generated by xa, xb, xc, ya, yb, yc, za, zb, and zc. R is quasi-Gorenstein, but $\dim R = 3$ and $\operatorname{depth} R = 2$, so R is not Cohen-Macaulay, hence not Gorenstein. Such an (obtuse) example is the result of a construction using degree products.

Lemma 1.10.46. *If $(R, \mathfrak{m})$ is Gorenstein and $\dim R = d$ (so $\omega_R \cong R[d]$) and $f \in \mathfrak{m}$ is a regular element, then ${}^R\!/\!_f$ is Gorenstein, and*

$$\omega^\bullet_{R/f} \cong {}^{\omega_R}\!/\!_{f\omega_R}[d-1] \cong \operatorname{Ext}^1\left({}^R\!/\!_f, \omega_R[d]\right).$$

Proof. We have the short exact sequence

$$0 \to R \xrightarrow{\cdot f} R \to {}^R\!/\!_{fR} \to 0.$$

After applying $\mathbf{R}\operatorname{Hom}(-, \omega_R^\bullet) = \mathbf{R}\operatorname{Hom}(-, \omega_R[d])$, we get

$$\mathbf{R}\operatorname{Hom}\left({}^R\!/\!_f, \omega_R[d]\right) \to \mathbf{R}\operatorname{Hom}(R, \omega_R[d]) \xrightarrow{\cdot f} \mathbf{R}\operatorname{Hom}(R, \omega_R[d]) \xrightarrow{+1} \mathbf{R}\operatorname{Hom}\left({}^R\!/\!_f, \omega_R[d]\right),$$

an exact triangle. Taking cohomology, we get the exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \omega_R \arrow[r, "\cdot f"] \arrow[d, "\wr"] & \omega_R \arrow[r] \arrow[d, "\wr"] & \operatorname{Ext}^1\left({}^R\!/\!_f, \omega_R\right) \arrow[r] & 0 \arrow[d, equal] \\
\operatorname{Hom}\left({}^R\!/\!_f, R\right) \arrow[d, "\wr"] & \operatorname{Hom}(R, \omega_R) & \operatorname{Hom}(R, \omega_R) & & \operatorname{Ext}^1(R, \omega_R) \\
\operatorname{Hom}\left({}^R\!/\!_f, \omega_R[d]\right) & & & &
\end{tikzcd}$$

We thus observe $\operatorname{Ext}^1\left({}^R\!/\!_f, \omega_R\right) \cong {}^{\omega_R}\!/\!_{f\omega_R}$, dimension shift by necessity, and $\omega^\bullet_{R/f} \cong \operatorname{Ext}^1\left({}^R\!/\!_f, \omega_R\right)$ by definition. ☐

Remark 1.10.47. For any complete local ring $(R, \mathfrak{m}, k)$ of dimension d with dualizing complex $\omega_R^\bullet$ and $E = E_R(k)$, we have

$$\operatorname{Hom}(H^d_{\mathfrak{m}}(R), E) \cong \operatorname{Ext}^{-d}(R, \omega_R^\bullet) \cong h^{-d}(\omega_R^\bullet) \cong \omega_R.$$

Corollary 1.10.48 (A corollary to local duality)**.** *Let $(R, \mathfrak{m})$ be a local domain. If $i < \dim R$, then $\operatorname{Ann} H^i_{\mathfrak{m}}(R) \neq 0$; i.e., there exists $c \neq 0$ such that $cH^i_{\mathfrak{m}}(R) = 0$.*

Proof. Let $\omega_R^\bullet$ be the normalized dualizing complex of R. Without loss of generality, we can assume R is complete. It suffices to find $c \neq 0$ so that $ch^{-i}(\omega_R^\bullet) = 0$. This is sufficient by Matlis/local duality, since as $\omega_R^\bullet$ is a dualizing complex, $\mathbf{R}\operatorname{Hom}(\omega_R^\bullet, \omega_R^\bullet) \cong R$ and

$$\omega_R^\bullet \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(\omega_R^\bullet, \omega_R^\bullet), \omega_R^\bullet) \cong \mathbf{R}\operatorname{Hom}(R, \omega_R^\bullet).$$

Therefore

$$ch^{-i}(\omega_R^\bullet) \cong ch^{-i}(\mathbf{R}\operatorname{Hom}(R, \omega_R^\bullet)) \cong ch^{-i}(\operatorname{Hom}(\mathbf{R}\Gamma_{\mathfrak{m}}(R), E)) \cong \operatorname{Hom}(ch^i(\mathbf{R}\Gamma_{\mathfrak{m}}(R)), E).$$

Recall that Matlis duality is faithful, by **Theorem 1.10.19 [Matlis]**. Thus $ch^{-i}(\omega_R^\bullet) = 0$ will force $(cH^i_{\mathfrak{m}}(R))^\vee = 0$, and therefore $cH^i_{\mathfrak{m}}(R) = 0$.

Finally, note that $h^{-i}(\omega_R^\bullet)$ is finitely generated, so set $K = \operatorname{Frac} R$ and localize $\omega_R^\bullet$. We get a complex $\omega_K^\bullet$ which is supported only in degree $-\dim R$. Thus, the localization of $h^{-i}(\omega_R^\bullet)$ is 0 for $i < \dim R$. $\square$

Remark 1.10.49. Recall in **Theorem 1.9.6**, we made the unproven claim that **Corollary 1.10.48** now takes care of.

Lemma 1.10.50. *If $(R, \mathfrak{m})$ is a noetherian local ring with a dualizing complex $\omega_R^\bullet$, then the complex $\mathbf{R}\operatorname{Hom}_R(F^e_*R, \omega_R^\bullet)$ is a dualizing complex for F^e_*R. Call this complex $\omega^\bullet_{F^e_*R}$, and one has that $\omega^\bullet_{F^e_*R} \cong F^e_*\omega_R^\bullet$.*

Definition 1.10.51 (trace of Frobenius)**.** The dual to $R \xrightarrow{F^e} F^e_*R$ is

$$\begin{array}{ccc} \operatorname{Hom}(F^e_*R, \omega_R) & \longrightarrow & \operatorname{Hom}(R, \omega_R) \\ \wr\| & & \wr\| \\ F^e_*\omega_R & \xrightarrow{\quad T^e \quad} & \omega_R \end{array}$$

which is called the **trace of Frobenius**, T^e.

Lemma 1.10.52. *For a quasi-Gorenstein ring R, after identifying $R \cong \omega_R$,*

$$\begin{array}{cccc} T^e: & F^e_*\omega_R & \longrightarrow & \omega_R \\ & \wr\| & & \wr\| \\ & F^e_*R & \longrightarrow & R \quad \in \operatorname{Hom}(F^e_*R, R) \end{array}$$

generates $\operatorname{Hom}_R(F^e_*R, R)$ *as a* F^e_*R*-module.*

Proof. First note that $\operatorname{Hom}(F^e_*R, R) \cong \operatorname{Hom}(F^e_*R, \omega_R)$ is a canonical module for F^e_*R. (This is okay, as F^e_*R is a finitely-generated R-module; i.e., R is F-finite, as we have been tacitly assuming throughout semester 1.) Note though that $F^e_*R \cong R$ as a ring; i.e., F^e_* is quasi-Gorenstein. Therefore $\operatorname{Hom}(F^e_*R, R)$ is cyclic as a F^e_*R-module. Set $\Phi^e \in \operatorname{Hom}_R(F^e_*R, R)$ a generator. Write $T^e(-) = \Phi^e(F^e_*d \cdot -)$ for some $d \in R$. Take duals to see that $F^e = F^e_*d \cdot (\Phi^e)^\vee$, but note that $F^e(1) = F^e_*1 = F^e_*d(\Phi^e)^\vee(1)$, which forces F^e_*d to be a unit in F^e_*R. $\square$

Example 1.10.53. Let $S = k[x_1, ..., x_d]$. Up to unit,

$$T^e = \begin{cases} F^e_*{x_1}^{p^e-1} \cdots {x_d}^{p^e-1} \mapsto 1; \\ \text{other monomials} \mapsto 0. \end{cases}$$

Lemma 1.10.54. *A quasi-Gorenstein local ring $(R, \mathfrak{m})$ that is F-injective is F-split.*

Proof. For a local ring $(R, \mathfrak{m})$, R is F-injective (i.e., $H^i_\mathfrak{m}(R) \hookrightarrow F^e_* H^i_\mathfrak{m}(R)$) if and only if

$$(H^i_\mathfrak{m}(R))^\vee \twoheadleftarrow F^e_*(H^i_\mathfrak{m}(R))^\vee.$$

Furthermore,

$$F^e_* h^{-i}(\omega^\bullet_R) \cong F^e_* H^i_\mathfrak{m}(R)^\vee \to H^i_\mathfrak{m}(R)^\vee \cong h^{-i}(\omega^\bullet_R)$$

is surjective for all i. For $i = \dim R$, the trace map $F^e_*\omega_R \xrightarrow{T^e} \omega_R$ is surjective. Therefore, when R is quasi-Gorenstein, $\omega_R \cong R$, and hence $F^e_* R \xrightarrow{T^e} R$ is surjective. Therefore, R is F-split, as desired. $\square$

Remark 1.10.55. Let's update the diagram in **Remark 1.9.8** with our findings. In the F-finite setting, we have:

$$\begin{tikzcd}
F\text{-regular} \arrow[rr, Rightarrow, "\textbf{Theorem 1.9.6}"] \arrow[d, Rightarrow, "\textbf{Remark 1.9.7}"'] & & F\text{-rational} \arrow[d, Rightarrow, "\textbf{Theorem 1.8.4}"'] \\
F\text{-split} \arrow[rr, Rightarrow, "\textbf{Lemma 1.7.1}"] \arrow[dr, Rightarrow, "\textit{Proof of \textbf{Theorem 1.6.3 [Ma]}}"'] & & F\text{-injective} \arrow[ll, Rightarrow, bend right, "+\text{quasi-Gorenstein, } \textbf{Lemma 1.10.54}"'] \arrow[u, Rightarrow, bend right, "+F\text{-nilpotent, } \textbf{Theorem 1.8.15 [Srinivas-Takagi]}"'] \\
& \text{anti-nilpotent} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Remark 1.10.56. Are we able to reverse the implication F-regular implies F-rational? Recall that a local ring $(R, \mathfrak{m})$ is F-rational if R is Cohen-Macaulay and $H^{\dim R}_\mathfrak{m}(R)$ has no proper F-stable submodules. Suppose R is any ring and $M \hookrightarrow H^{\dim R}_\mathfrak{m}(R)$ is an F-stable submodule. Taking duals, we get

$$\operatorname{Hom}(H^{\dim R}_\mathfrak{m}(R), E) \cong \omega_R \to M^\vee = \operatorname{Hom}(M, E).$$

As M is F-stable, $M^\vee$ is "T-stable;" that is,

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, dashed, "F"] & H^{\dim R}_\mathfrak{m}(R) \arrow[d, "F"] \\
M \arrow[r, hook] & H^{\dim R}_\mathfrak{m}(R)
\end{tikzcd} \quad \text{implies via Matlis that} \quad \begin{tikzcd}
\omega_R \arrow[r, two heads] & M^\vee \\
\omega_R \arrow[r, two heads] \arrow[u, "T"] & M^\vee \arrow[u, dashed, "T"']
\end{tikzcd}$$

Set $N = \ker(\omega_R \to M^\vee)$. We get the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, dashed, "T"] & \omega_R \arrow[r] \arrow[d, "T"] & M^\vee \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & N \arrow[r] & \omega_R \arrow[r] & M^\vee \arrow[r] & 0
\end{tikzcd}$$

That is, $T(N) \subseteq N$, so $\ker(\omega_R \to M^\vee)$ is honestly T-stable, as it is a submodule.

Conversely, any T-stable submodule $N \subseteq \omega_R$ has a cokernel

$$0 \to N \to \omega_R \to \omega_R/N \to 0.$$

Since Matlis dual is fully faithful, $\omega_R/N \cong M^\vee$ for some module M. Taking duals again, we get $M \subseteq H^{\dim R}_\mathfrak{m}(R)$ is F-stable.

Therefore, Matlis duality induces a bijection

$$\{N \subseteq \omega_R \ T\text{-stable}\} \xleftrightarrow{\operatorname{Hom}(-,E)} \{M \subseteq H^{\dim R}_\mathfrak{m}(R) \ F\text{-stable}\}.$$

Theorem 1.10.57. *A Gorenstein F-rational domain is F-regular.*

Proof. Let R be a Gorenstein F-rational domain. (Note that as F-rational rings are Cohen-Macaulay, R must be Gorenstein, not quasi-Gorenstein.) Fix $c \neq 0$. Assume for the sake of contradiction that there is no splitting for $R \to F_*^e R \cong R^{\frac{1}{p^e}}$, $1 \mapsto F_*^e c = c^{\frac{1}{p^e}}$. Now consider the set

$$\mathfrak{a} = \left(\varphi\left(c^{\frac{1}{p^e}}\right) \mid \varphi \in \operatorname{Hom}\left(R^{\frac{1}{p^e}}, R\right), e \in \mathbf{N}\right),$$

which is the ideal in R generated by $\varphi(c^{\frac{1}{p^e}})$ for all φ. Note that the nonsplitting of $R \to R^{\frac{1}{p^e}}$, $1 \mapsto c^{\frac{1}{p^e}}$, is equivalent to $\mathfrak{a} \neq R$. Also note that $\mathfrak{a}$ is F-stable, and $\mathfrak{a} \neq 0$. Since R is Gorenstein, $R \cong \omega_R$. Under this identification, ω_R has a nonzero proper T-stable submodule, which contradicts **Remark 1.10.56**, since R is F-rational. $\square$

Remark 1.10.58. Once more, update the diagram in **Remark 1.10.55**. We have:

$$\begin{tikzcd}
F\text{-regular} \arrow[rr, Rightarrow, "\textbf{Theorem 1.9.6}"] \arrow[d, Rightarrow, "\textbf{Remark 1.9.7}"'] & & F\text{-rational} \arrow[ll, Rightarrow, bend right, "+\text{Gorenstein, }\textbf{Theorem 1.10.57}"'] \arrow[d, Rightarrow, "\textbf{Theorem 1.8.4}"'] \\
F\text{-split} \arrow[rr, Rightarrow, "\textbf{Lemma 1.7.1}"] \arrow[dr, Rightarrow, "\textit{Proof of } \textbf{\textit{Theorem 1.6.3 [Ma]}}"'] & & F\text{-injective} \arrow[ll, Rightarrow, bend right, "+\text{quasi-Gorenstein, }\textbf{Lemma 1.10.54}"'] \arrow[u, Rightarrow, bend right, "+F\text{-nilpotent, }\textbf{Theorem 1.8.15 [Srinivas-Takagi]}"'] \\
& \text{anti-nilpotent} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Example 1.10.59. In particular, if S is a polynomial ring, then S/f is Gorenstein. One can check that S/f is F-injective or F-rational via Fedder-type statements.

Remark 1.10.60. Let's now see another proof of **Theorem 1.7.11** that uses dualizing complexes. The one that follows is more homological, as it is element free. We will be able to use it to prove that F-rationality deforms.

Theorem 1.10.61. *Let $(R, \mathfrak{m})$ be Cohen-Macaulay. Let R have a dualizing "complex" $\omega_R^\bullet$. Let $f \in \mathfrak{m}$ be a regular element. If R/fR is F-injective, then R is F-injective.*

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot f"] \arrow[d, "f^{p^e-1}F^e"] & R \arrow[r] \arrow[d, "F^e"] & R/f \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & F_*^e R \arrow[r] & F_*^e R \arrow[r] & F_*^e R/f \arrow[r] & 0
\end{tikzcd}$$

As R is Cohen-Macaulay, $\omega_R^\bullet \cong_q \omega_R[\dim R]$, where ω_R is a canonical module. If we apply $\operatorname{Hom}(-, \omega_R)$, we first get the long exact sequence

$$\cdots \to \operatorname{Hom}\left(R/f, \omega_R\right) \to \operatorname{Hom}(R, \omega_R) \xrightarrow{\cdot f} \operatorname{Hom}(R, \omega_R) \to \operatorname{Ext}^1\left(R/f, \omega_R\right) \to \operatorname{Ext}^1(R, \omega_R) \to \cdots.$$

Now, ω_R is torsion-free, so $\operatorname{Hom}\left(R/f, \omega_R\right) = 0$. Furthermore, we have that $\operatorname{Hom}(R, \omega_R) \cong \omega_R$, that $\operatorname{Ext}^1\left(R/f, \omega_R\right) \cong \omega_{R/f} \cong \omega_R/f\omega_R$, and that $\operatorname{Ext}^1(R, \omega_R) = 0$. We obtain a diagram

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & \operatorname{coker} T^e \arrow[u] \arrow[r] & D \arrow[u] \arrow[r] & 0 & \\
0 \arrow[r] & \omega_R \arrow[u] \arrow[r] & \omega_R \arrow[u] \arrow[r] & \omega_{R/f} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F^e_* \omega_R \arrow[u, "T^e_R"] \arrow[r] & F^e_* \omega_R \arrow[u, "F^e_* f^{p^e-1} T^e_R"] \arrow[r] & F^e_* \omega_R / f\omega_R \arrow[u, "T^e_{R/f}"] \arrow[r] & 0
\end{tikzcd}$$

To show R is F-injective, it suffices to show that T^e_R is surjective. Since R/f is F-injective, $T^e_{R/f}$ is surjective. We have a surjection $\mu : \operatorname{coker} T^e \twoheadrightarrow D$ which comes from the Snake Lemma. Assume for the sake of contradiction that $\operatorname{coker} T^e \neq 0$. Write $\operatorname{coker} T^e \cong \omega_R / T^e(F^e_* \omega_R)$, and $D \cong \omega_R / T^e(F^e_* f^{p^e-1} \omega_R)$. There is a natural map $\eta : D \to C$ so that

$$C \xrightarrow{\mu} D \xrightarrow{\eta} C$$

is the multiplication by f map. Thus $fC \cong C$, contradicting **Lemma 1.2.9 [Nakayama's Lemma]**. □

Theorem 1.10.62 (Smith)**.** *F-rational deforms. That is, let $(R, \mathfrak{m})$ be an F-rational ring with a dualizing complex $\omega^\bullet_R$, and let $f \in \mathfrak{m}$ be a regular element. If R/f is F-rational, then R is F-rational.*

Proof. First, note that R/f is Cohen-Macaulay, so R is also Cohen-Macaulay. Next, we need to show that $H^d_\mathfrak{m}(R)$ has no proper F-stable submodules. By **Remark 1.10.56**, there is a correspondence between F-stable submodules of $H^d_\mathfrak{m}(R)$ and T-stable submodules of ω_R.

Set $\tau(\omega_R)$ to be "the smallest" nonzero T-stable submodule. Similarly, set $\tau(\omega_{R/f})$ to be the smallest nonzero T-stable submodule. We will call $\tau(\omega_R)$ the test submodule of ω_R and prove the following claim later, where such a c is called the test element:

> **Claim.** *There is a regular element $c \in R \setminus \{0\}$ such that*
>
> $$\sum_e T^e\left(F^e_* c\omega_R\right) = \tau\left(\omega_R\right).$$
>
> *In fact, we can pick c simultaneously so that c is a test element for both ω_R and $\omega_{R/f}$.*

Assume this claim, and consider the map $R \to F^e_* R$ defined by $1 \mapsto F^e_* c$. This induces the following diagram for every e.

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot f"] \arrow[d] & R \arrow[r] \arrow[d] & R/f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r] & F^e_* R \arrow[r] & F^e_* R/f \arrow[r] & 0
\end{tikzcd}$$

$$1 \mapsto F^e_* f^{p^e-1} c, \qquad 1 \mapsto F^e_* c, \qquad 1 \mapsto [F^e_* c]$$

Adding together, we get by applying $\operatorname{Hom}(-, \omega_R)$ to all diagrams

$$\begin{tikzcd}
0 \arrow[r] & \omega_R \arrow[r] & \omega_R \arrow[r] & \omega_{R/f} \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_e F^e_* \omega_R \arrow[u, "\alpha"] \arrow[r] & \bigoplus_e F^e_* \omega_R \arrow[u, "\beta"] \arrow[r] & \bigoplus_e F^e_* \omega_{R/f} \arrow[u, "\gamma"] \arrow[r] & 0
\end{tikzcd}$$

where α is the dual of $1 \mapsto F^e_* c$, β is the dual of $1 \mapsto F^e_* f^{p^e-1}c$, and γ is the dual of $1 \mapsto [F^e_* c]$.

The goal is to show that $\tau(\omega_R) = \omega_R$, forcing a contradiction. To that end, see that $\operatorname{im}\alpha = \tau(\omega_R)$, $\operatorname{im}\gamma = \tau(\omega_{R/f})$, and $\operatorname{im}\beta \subseteq \tau(\omega_R)$. By assumption, $\tau(\omega_{R/f}) = \omega_{R/f}$, since R/f is F-rational.

Set $C = \operatorname{coker}\alpha$ and $D = \operatorname{coker}\beta$. By the Snake Lemma, there is a map $\mu : C \twoheadrightarrow D$. There is a natural map $\eta : D \to C$. One can deduce that $C = 0$ by **Lemma 1.2.9 [Nakayama's Lemma]**. Roughly, $\eta\mu$ is multiplication by fc, and $fc \in \mathfrak{m}$. ☐

Remark 1.10.63. We have also proved **Theorem 1.10.62 [Smith]** using a technique that doesn't use duality. Recall **Theorem 1.8.18**.

Theorem 1.10.64 (Singh)**.** *Let $n, m \in \mathbf{Z}$ with $m - \frac{m}{n} > 2$. Let*

$$R = k[A,B,C,D,T]/\mathfrak{a},$$

where $\mathfrak{a}$ is the 2×2 minors of

$$\begin{bmatrix} A^2+T^m & B & D \\ C & A^2 & B^n - D \end{bmatrix}.$$

The ring R/tR is F-regular, but R is not F-regular. Hence, F-regular does not deform in general.

Remark 1.10.65. We have the following deformation results:

- F-regular rings do not deform in general. (**Theorem 1.10.64 [Singh]**.)
- F-split rings do not deform in general. (**Theorem 1.7.18 [Singh]**.)
- F-rational rings deform. (**Theorem 1.10.62 [Smith]**.)
- F-injective rings deform, when the ring is Cohen-Macaulay+ (like, for instance, Cohen-Macaulay at all primes other than the maximal ideal (the punctured spectrum)). (**Theorem 1.7.11 [Fedder]**.) It is conjectured that all F-injective rings deform.

Using the diagram in **Remark 1.10.58**, we see that Gorenstein F-regular rings deform.

Theorem 1.10.66 (Shimomoto-Taniguchi-Tavanfar)**.** *Let $(R, \mathfrak{m})$ be a local noetherian ring of dimension d. Let $f \in \mathfrak{m}$ be a regular element. If R/f is quasi-Gorenstein and the Frobenius action on $H^{d-1}_{\mathfrak{m}}\left(R/f\right)$ is injective, then R is quasi-Gorenstein.*

Proof. Recall in **Theorem 1.7.19** that proving R/f is F-injective implies R is F-split used the identification of f as a surjective element.

Without loss of generality, assume that R is complete. The first step is to establish that f is a surjective element; i.e.,

$$H^i_{\mathfrak{m}}\left(R/f^nR\right) \to H^i_{\mathfrak{m}}\left(R/fR\right)$$

is surjective for all i. By a diagram chase, this occurs if and only if $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^i_{\mathfrak{m}}(R)$ is surjective. (The proof of this is an exercise in direct limits and local cohomology.)

Let's check that this is enough. Using the short exact sequence

$$0 \to R \xrightarrow{\cdot f} R \to R/fR \to 0,$$

we get the long exact sequence

$$\cdots \to H^{d-1}_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^{d-1}_{\mathfrak{m}}(R) \to H^{d-1}_{\mathfrak{m}}\left(R/fR\right) \to H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^d_{\mathfrak{m}}(R) \to 0.$$

Assuming that f is a surjective element, we get the short exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}\left(R/fR\right) \to H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^d_{\mathfrak{m}}(R) \to 0.$$

Take Matlis duals to get

$$0 \to \omega_R \xrightarrow{\cdot f} \omega_R \to \omega_{R/f} \cong \omega_R/f\omega_R \to 0.$$

Since R/fR is quasi-Gorenstein, $\omega_{R/f} \cong R/fR$, so ω_R is cyclic. Write $\omega_R = R/J$ for some ideal J. The goal is then to show that $J = 0$. What follows is a sketch.

Since R/fR is quasi-Gorenstein, R is unmixed. There is an older result by Aoyama that says ω_R is faithful; i.e., $\operatorname{Ann}\omega_R = 0$. Thus, $J = 0$, and therefore $\omega_R \cong R$.

Next, we can focus on showing that f is indeed a surjective element. By assumption, the map

$$H^{d-1}_{\mathfrak{m}}\left(R/fR\right) \to F^e_* H^{d-1}_{\mathfrak{m}}\left(R/fR\right)$$

is injective, so it dualizes to a surjective map

$$\begin{array}{ccc} F^e_*\omega_{R/f} & \longrightarrow & \omega_{R/f} \\ \wr\wr & & \wr\wr \\ F^e_* R/fR & \longrightarrow & R/fR \end{array}$$

That is, R/fR is F-split. This forces f to be a surjective element by the proof of **Theorem 1.7.19**. $\square$

1.11 Frobenius Operators

Remark 1.11.1. An easy, but fundamental, observation of Lyubeznik and Smith is the following. Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring.

$$E_R(k) \cong H^{\dim R}_{\mathfrak{m}}(R)^\vee$$

The natural Frobenius map on $H^{\dim R}_{\mathfrak{m}}(R)$ gives a natural "Frobenius operator" on $E_R(k)$; that is, an $R\{F\}$-structure.

Definition 1.11.2 (set of Frobenius operators)**.** Fix an R-module M. Set

$$\mathcal{F}^e(M) = \left\{\rho : M \to M \mid \rho \in \operatorname{Hom}_R(M, F^e_* M), \rho \text{ is a } p^e\text{-linear map}\right\}.$$

Call $\mathcal{F}^e$ the **set of Frobenius operators of order** e.

Definition 1.11.3 (ring of Frobenius operators)**.** We can patch the set of Frobenius operators of all orders together; we get a noncommutative graded ring

$$\mathcal{F}(M) = \mathcal{F}^0(M) \oplus \mathcal{F}^1(M) \oplus \mathcal{F}^2(M) \oplus \cdots.$$

That is, given a p^e-linear map $\rho : M \to M$ and a $p^{e'}$-linear map $\rho' : M \to M$, then both $\rho \circ \rho'$ and $\rho' \circ \rho$ are $p^{e+e'}$-linear. One calls $\mathcal{F}(M)$ the **ring of Frobenius operators**.

Theorem 1.11.4 (Lyubeznik-Smith)**.** *If $(R, \mathfrak{m}, k)$ is a complete local Gorenstein ring with $E = E_R(k)$, then $\mathcal{F}(E)$ is finitely generated over $\mathcal{F}^0(E)$.*

Remark 1.11.5. The Gorenstein assumption is necessary; if

$$R = k\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \Big/ I_2$$

where I_2 is the 2×2 minors of

$$\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix},$$

then $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$.

Theorem 1.11.6 (Katzman-Schwede-Singh-Zhang)**.** *Let $S = k[x_1, ..., x_d]$. Let $n \in \mathbf{N}$. Set $\mathcal{M}$ to be the set of monomials of degree n in the variables x_i, $i \in \{1, ..., d\}$. That is,*

$$\mathcal{M} = \left\{ {x_1}^{i_1} \cdots {x_d}^{i_d} \mid \sum_{j=1}^{d} i_j = n \right\}.$$

The ring $R = k\widehat{[m \mid m \in \mathcal{M}]} \subseteq \widehat{S}$ (which is called the n-Veronese subalgebra of S) satisfies the following.

1. *If $M = R_{x_1 \cdots x_d}$ and N is the submodule of M generated by ${x_1}^{i_1} \cdots {x_d}^{i_d}$ with $i_\ell \geq 1$ for some ℓ, then $E_R(k) \cong {}^{M}\!/\!_{N}$.*
2. *$\mathcal{F}^e(E)$ is generated over $\mathcal{F}^0(E)$ by $\frac{1}{{x_1}^{a_1} \cdots {x_d}^{a_d}} F^e$ with $a_i \leq p^e - 1$ and $\sum a_i \equiv 0 \mod n?p?$.*

Proof sketch. The proof relies on a delicate identification. It was known from before (in Blickle's thesis) that, if $R = {}^{S'}\!/\!_{\mathfrak{a}}$ with $S' = k[\![z_1, ..., z_d]\!]$, then

$$\mathcal{F}^e(E) \cong {}^{\left(\mathfrak{a}^{[p^e]} :_{S'} \mathfrak{a}\right)}\!\Big/\!_{\mathfrak{a}^{[p^e]}}$$

(Recall **Corollary 1.4.24 [Fedder's Criterion]**.)

However, one needs to understand the operation in the graded ring. Write

$$\mathcal{A} = \bigoplus_{n \in \mathbf{N}} [\mathcal{A}]_n$$

where $[\mathcal{A}]_n$ are the degree n parts of $\mathcal{A}$. Thus, $\mathcal{A}$ is an $\mathbf{N}$-graded algebra. Thus $[\mathcal{A}]_{\leq n}[\mathcal{A}]_{\leq m} \subseteq [\mathcal{A}]_{\leq n+m}$.

Define

$$\mathcal{T}(\mathcal{A}) = \bigoplus_{e \geq 0} [\mathcal{A}]_{p^e - 1}.$$

Therefore $[\mathcal{T}(\mathcal{A})]_e = [\mathcal{A}]_{p^e-1}$. Give $\mathcal{T}(\mathcal{A})$ the noncommutative operation

$$f * g = f g^{p^e} \in \mathcal{A}$$

for $f \in [\mathcal{T}(\mathcal{A})]_e$ and $g \in [\mathcal{T}(\mathcal{A})]_{e'}$. One can check that

$$f * g \in [\mathcal{A}]_{(p^e-1)+p^e(p^{e'}-1)} = [\mathcal{A}]_{p^{e+e'}-1} = [\mathcal{T}(\mathcal{A})]_{e+e'}.$$

The theorem follows by writing $\mathcal{F}(E) \cong \mathcal{T}(\mathcal{A})$ for some graded algebra $\mathcal{A}$ which is based on symbolic powers. Roughly, under "nice" assumptions, ω_R is isomorphic to a height 1 ideal of R. One can promote the set of all height 1 ideals to a group, called the divisor class group. The powers in this group of ω_R are the symbolic powers ${\omega_R}^{(n)}$. Here, ${\omega_R}^{(-n)}$ for $n \geq 0$ is $\mathrm{Hom}_R({\omega_R}^{(n)}, R)$. The algebra needed for the theorem is therefore

$$\mathcal{A} = \bigoplus_{n \geq 0} {\omega_R}^{(-n)},$$

which is called the anticanonical algebra of R. $\square$

Remark 1.11.7. The anticanonical algebra $\mathcal{A}$ is almost never noetherian, but it is noetherian when R is Gorenstein, or when ω_R has torsion in the class group.

Example 1.11.8. For $n = 3$ and $d = 2$, let $R = k[\![x^3, x^2y, xy^2, y^3]\!] \subseteq S = k[\![x, y]\!]$. For a fixed e, we have the following.

1. If $p \equiv 1 \mod 3$, then

$$\mathcal{F}(E) = R\left\{ \frac{1}{(xy)^{p-1}} F \right\},$$

which is finitely generated over $\mathcal{F}^0(E)$.

2. If $p \equiv 2 \bmod 3$, then

$$\mathcal{F}(E) = R\left\{\frac{1}{x^{p-3}y^{p-1}}F, \frac{1}{x^{p-2}y^{p-2}}F, \frac{1}{x^{p-1}y^{p-3}}F, \frac{1}{x^{p^2-1}y^{p^2-1}}F^2\right\},$$

which is finitely generated over $\mathcal{F}^0(E)$.

3. If $p = 3$, then

$$\mathcal{F}(E) = R\left\{\frac{1}{xy^2}F, \frac{1}{x^2y}F, \frac{1}{x^7y^8}F^2, \frac{1}{x^8y^7}F^2, ..., \frac{1}{x^{25}y^{26}}F^3, \frac{1}{x^{26}y^{25}}F^3, ...\right\}$$

is not finitely generated.

Definition 1.11.9 (**Q**-Gorenstein)**.** Call a ring R **Q-Gorenstein** if ${\omega_R}^{(n)}$ is principal for $n > 0$.

Remark 1.11.10. Theorem 1.11.6 [Katzman-Schwede-Singh-Zhang] says that under these nice conditions, $\mathcal{F}(E)$ is finitely generated when R is **Q**-Gorenstein with ${\omega_R}^{(n)}$ principal such that p does not divide n.

Remark 1.11.11. Via duality, one can show that $\mathrm{Hom}_R({\omega_R}^{(p^e-1)}, R) \cong {\omega_R}^{(1-p^e)}$, and identify

$$\mathcal{F}(E) \cong \bigoplus_{e \geq 0} {\omega_R}^{(1-p^e)} F^e.$$

This justifies the strange operation on $\mathcal{T}(\mathcal{A})$ and gives it an explicit form:

$$(aF^e) \circ (bF^{e'}) = ab^{p^e} F^{e+e'}.$$

1.12 FFRT Rings

Definition 1.12.1 (finite F-representation type)**.** For a ring $(R, \mathfrak{m})$, we say that R has **finite F-representation type (FFRT)** if there is a finite collection $N_1, ..., N_s$ of R-modules such that $F_*^e R \cong {N_1}^{\oplus a_1} \oplus \cdots \oplus {N_s}^{\oplus a_s}$ for integers $a_i \in \mathbf{N}$. Note that N_i does not depend on e, while a_i does.

Remark 1.12.2. Recall our tacit assumption that R is F-finite; i.e., $F_*^e R$ is finitely generated for all e. Thus if R has FFRT, each N_i will also be finitely generated.

Example 1.12.3. If R is a regular ring, then by **Theorem 1.1.24 [Kunz]**, $F_*^e R \cong R^{\oplus e \dim R}$. Hence R has FFRT; $N_1 = R$ and $a_1(e) = e \dim R$.

Example 1.12.4. Any direct summand of a regular ring has FFRT. In particular, the n^{th} Veronese has FFRT for $n \not\equiv 0 \bmod p$.

Example 1.12.5. Any artinian local ring (which must have finite length) has FFRT.

Example 1.12.6. Any monomial quotient of a polynomial ring has FFRT. That is, if $R = {}^{S}\!/_{\mathfrak{a}}$ where $S = k[x_1, ..., x_d]_{\mathfrak{m}}$ and $\mathfrak{a}$ is generated by monomials, then R has FFRT.

Example 1.12.7. The ring

$$R = {}^{k\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}\!\Big/_{I_2}$$

where I_2 is the 2×2 minors of

$$\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$$

has FFRT.

Remark 1.12.8. It is an open question if the ring

$$R = \left. k\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \middle/ \det \right.$$

has FFRT.

Remark 1.12.9. When R is Cohen-Macaulay, F^e_* is a Cohen-Macaulay R-module of dimension $d = \dim R$. Any direct summand N of $F^e_* R$ is also a Cohen-Macaulay R-module of dimension d.

Definition 1.12.10 (maximal Cohen-Macaulay module)**.** We call a Cohen-Macaulay R-module of dimension $\dim R$ a **maximal Cohen-Macaulay module (MCM)**.

Definition 1.12.11 (finite Cohen-Macaulay type)**.** A Cohen-Macaulay ring R with finitely many indecomposible MCMs is said to have **finite Cohen-Macaulay type**.

Remark 1.12.12. If R has finite Cohen-Macaulay type, then R has FFRT.

Theorem 1.12.13 (Hochster-Nuñez-Betancourt, Dao-Quy)**.** *If R has FFRT, then for each i and ideal $\mathfrak{a}$, $H^i_{\mathfrak{a}}(R)$ has finitely many associated primes.*

Proof. Set $M = H^i_{\mathfrak{a}}(R)$. Recall that $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if $H^0_{\mathfrak{p}}(M_{\mathfrak{p}}) \neq 0$. As F^e_*- commutes with localization, it's easy to check that $\operatorname{Ass}(M) = \operatorname{Ass}(F^e_* M)$. Now, it's clear that

$$\operatorname{Ass} H^i_{\mathfrak{a}}(R) \subseteq \bigcup_e \operatorname{Ass} H^i_{\mathfrak{a}}(F^e_* R) \subseteq \bigcup_{j=1}^s \operatorname{Ass} H^i_{\mathfrak{a}}(N_j).$$

Thus

$$\left|\operatorname{Ass} H^i_{\mathfrak{a}}(R)\right| \leq \sum_{j=1}^s \left|\operatorname{Ass} H^i_{\mathfrak{a}}(N_j)\right| < \infty,$$

as desired. $\square$

Remark 1.12.14. In the same paper, Hochster and Nuñez-Betancourt also prove that

$$R = \left. k\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \middle/ \det \right.$$

satisfies the fact that each $H^i_{\mathfrak{a}}(R)$ has only finitely many associated primes.

Remark 1.12.15. The first proof of the associated prime theorem was from Takagi-Takahashi, in the quasi-Gorenstein case. They proved that

$$\operatorname{Ass} H^i_{\mathfrak{a}}(\omega_R) \subseteq \bigcup_j \operatorname{Ass} \operatorname{Ext}^i\left(N_j / \mathfrak{a} N_j, \omega_R \right).$$

Indeed, assume $F^e_* R \cong {N_1}^{\oplus a_1} \oplus \cdots \oplus {N_s}^{\oplus a_s}$. We get that

$$F^e_* \mathbf{R}\operatorname{Hom}_R\left(R/\mathfrak{a}^{[p^e]}, \omega^\bullet_R \right) \cong \mathbf{R}\operatorname{Hom}_{F^e_* R}\left(F^e_* R / \mathfrak{a} F^e_* R, F^e_* \omega^\bullet_R \right) \cong \mathbf{R}\operatorname{Hom}_R\left(F^e_* R / \mathfrak{a} F^e_* R, \omega^\bullet_R \right),$$

where the second isomorphism is a quite technical application of duality. Commuting the limit, we then get

$$\mathbf{R}\operatorname{Hom}_R\left(F^e_* R / \mathfrak{a} F^e_* R, \omega^\bullet_R \right) \cong \bigoplus_j \mathbf{R}\operatorname{Hom}_R\left(N_j / \mathfrak{a} N_j, \omega^\bullet_R \right).$$

One can then take cohomology to get

$$F^e_* \operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, \omega_R \right) \cong \bigoplus_j F^e_* \operatorname{Ext}^i\left(N_j / \mathfrak{a} N_j, \omega_R \right)^{\oplus a_j},$$

and

$$\lim_{e \to \infty} F^e_* \operatorname{Ext}^i\left(R/\mathfrak{a}^{[p^e]}, \omega_R \right) \cong H^i_{\mathfrak{a}}(\omega_R).$$

Remark 1.12.16. A question: does FFRT imply F-regular?

1.13 Test Ideals

Remark 1.13.1. Recall that in **Theorem 1.10.62 [Smith]**, we assumed that for a local ring $(R, \mathfrak{m})$ with canonical module ω_R, we have the equality

$$\tau(\omega_R) = \sum_e T^e(F_*^e c\omega_R),$$

where T is the trace of Frobenius, $\tau(\omega_R)$ is the smallest nonzero T-stable submodule of ω_R, and $c \in R \setminus \{0\}$ is a test element. This claim will follow from a general theory of test ideals. Roughly, to check if R is an F-regular domain, one needs to check a priori an infinite family of conditions:

$$\text{For every } d \in R \setminus \{0\}, \text{ there exists } e = e(d) \gg 0 \text{ such that } R \to F_*^e R \xrightarrow{\cdot F_*^e d} F_*^e R \text{ splits.} \qquad (*_d)$$

However, we can make our lives much easier by finding a single $c \in R \setminus \{0\}$ so that if $(*_c)$ holds for all e, then $(*_d)$ holds for all d.

Definition 1.13.2 (compatible)**.** For a ring R, call an ideal $\mathfrak{a}$ **compatible** provided $\varphi(F_*^e\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ and all e.

Remark 1.13.3. Contrast the above definition with **Definition 1.4.16 [φ-compatible]**, where we fix one $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ and ask that $\varphi(F_*^e\mathfrak{a}) \subseteq \mathfrak{a}$ for this one φ. For any fixed $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ and $\mathfrak{a}$ which is a φ-compatible ideal, the map φ descends to a map $\overline{\varphi} \in \operatorname{Hom}_R\left(F_*^e R/\mathfrak{a}, R/\mathfrak{a}\right)$.

$$\begin{tikzcd}
F_*^e R \arrow[r, "\varphi"] \arrow[d, two heads] & R \arrow[d, two heads] \\
F_*^e\left(R/\mathfrak{a}\right) \arrow[r, "\overline{\varphi}"'] & R/\mathfrak{a}
\end{tikzcd}$$

A compatible ideal is φ-compatible for all possible φ.

Remark 1.13.4. In the literature, compatible ideals $\mathfrak{a}$ are also called "uniformly" compatible. In the case that $\mathfrak{a}$ is prime and compatible, $\mathfrak{a}$ is also called "F-pure centers."

Remark 1.13.5. There is a related class of singularities called F-pure. This is defined by asking that the Frobenius map $F : R \to F_*^e R$ is pure; i.e., that $\operatorname{id} \otimes F^e : M \to M \otimes F_*^e R$ is injective. In the F-finite case (which we have still been tacitly assuming), F-split is equivalent to F-pure.

Lemma 1.13.6. *The collection of compatible ideals in a ring is closed under finite sum and intersection. The minimal primes of a compatible ideal are compatible.*

Example 1.13.7. If R is regular, then there are no proper compatible ideals. Indeed, pick any $f \in R$ and extend $F_*^e f$ to a basis for $F_*^e R$, which is free by **Theorem 1.1.24 [Kunz]**. Let $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ be the projection onto the $F_*^e f$ factor. If $\mathfrak{a}$ is compatible and $f \in \mathfrak{a} \setminus \{0\}$, then

$$1 = \varphi(F_*^e f) \in \mathfrak{a},$$

so $\mathfrak{a} = R$.

Remark 1.13.8. The above argument works for F-regular rings too; we just don't need a basis.

Remark 1.13.9. Historically, a source of compatible ideals that were studied were of the form $\mathfrak{a} = \operatorname{Ann} N$ for $N \subseteq H^i_{\mathfrak{m}}(R)$ with N an F-stable submodule. Recall **Theorem 1.4.28 [Schwede]**; if R is F-split (F-pure), then there are only finitely many such ideals.

Remark 1.13.10. Compatibility is linked to splitting in the following way:

Suppose $\mathfrak{q} \in \operatorname{Spec} R$ is prime and compatible. For each $f \in \mathfrak{q}$, the map

$$R \to F_*^e R \xrightarrow{\cdot F_*^e f} F_*^e R$$

cannot split. Otherwise, set $\varphi : F^e_* R \to R$ the splitting and note that $1 = \varphi(F^e_* f) \in \mathfrak{q}$. Conversely, if $\mathfrak{q}$ is not compatible, then for some e, the local maps

$$R_{\mathfrak{q}} \to F^e_* R_{\mathfrak{q}} \xrightarrow{\cdot F^e_* f} F^e_* R_{\mathfrak{q}}$$

split. That is, we could have defined $\mathfrak{q}$ as compatible if for all $f \in \mathfrak{q}$, the map

$$R_{\mathfrak{q}} \to F^e_* R_{\mathfrak{q}} \xrightarrow{\cdot F^e_* f} F^e_* R_{\mathfrak{q}}$$

does not split.

Definition 1.13.11 (test ideal)**.** Fix a local ring $(R, \mathfrak{m})$. Define the **test ideal** of $(R, \mathfrak{m})$ to be the unique smallest nonzero compatible ideal, denoted $\tau(R)$.

☡ **Warning! 1.13.12.** There is no reason to assume that $\tau(R)$ exists! Even in the F-split case, where for each fixed $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ there are finitely many φ-compatible ideals, it is still possible that all ideals are not uniformly compatible.

Definition 1.13.13 (test element)**.** Elements of $\tau(R)$ are called **test elements**.

Example 1.13.14. By **Remark 1.13.19**, for any F-regular ring R, $\tau(R) = R$.

Example 1.13.15. For $R = {k[x, y, z]_{\mathfrak{m}}}/{(x^3 + y^3 + z^3)}$ and $p \equiv 1 \bmod 3$, $\tau(R) = \mathfrak{m}$, which we will later see.

Remark 1.13.16. Suppose $\tau(R)$ exists. Let $c \in \tau(R) \setminus \{0\}$. Note that the ideal

$$J = \sum_{e \geq 0} \sum_{\varphi \in \operatorname{Hom}_R(F^e_* R, R)} \varphi(F^e_* cR)$$

is compatible. Note also that $c \in J$ (taking $e = 0$ and $\varphi = \mathrm{id}$). In fact, J is the smallest ideal that is compatible and contains c; thus $J = \tau(R)$. Thus, to prove that $\tau(R)$ exists, it suffices to find one single element in $\tau(R) \setminus \{0\}$.

Theorem 1.13.17 (Existence of test elements)**.** *If R is reduced and $c \in R \setminus \{0\}$ such that R_c is F-regular (or just regular), then c has a power which is a test element. Furthermore, if there exists $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\varphi(F^e_* 1) = c$, then c^3 is a test element.*

Lemma 1.13.18. *Formation of the test ideal commutes with localization and completion.*

Proof. Let W be a multiplicatively closed set. Assume there exists $c \in R \setminus \{0\}$ such that $c \in \tau(R)$ and $\frac{c}{1} \in \tau(W^{-1}R)$. View both $\tau(W^{-1}R)$ and $W^{-1}\tau(R)$ as ideals in $W^{-1}R$. Note that by F-finiteness,

$$\operatorname{Hom}_{W^{-1}R}(F^e_* W^{-1}R, W^{-1}R) \cong \operatorname{Hom}_R(F^e_* R, R) \otimes_R W^{-1}R.$$

Thus any map $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ satisfies

$$\varphi \frac{(F^e_* c)}{1} = \frac{\varphi}{1} \left(F^e_* \frac{c}{1} \right).$$

Summing over all maps, one obtains $\tau(W^{-1}R) = W^{-1}\tau(R)$.

Completion is similar. □

Theorem 1.13.19. *A ring R is F-regular if and only if $\tau(R) = R$.*

Proof. As $\tau(R)$ commutes with localization by **Lemma 1.13.18**, it suffices to assume that R is local with maximal ideal $\mathfrak{m}$. If R is F-regular, then for any $c \in \tau(R) \setminus \{0\}$, there is a splitting $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ of the natural map

$$R \to F^e_* R \xrightarrow{\cdot F^e_* c} F^e_* R$$

so $1 = \varphi(F^e_* c) \in \tau(R)$. Thus, $\tau(R) = R$.

Conversely, if $\tau(R) = R$ and $c \in R \setminus \{0\}$, consider the sum

$$\sum_e \sum_\varphi \varphi(F^e_* cR) \neq 0,$$

which is compatible. Thus, $R = \tau(R) \subseteq \sum\sum \varphi(F^e_* cR)$. Therefore, there exists e so that $\varphi(F^e_* c) \notin \mathfrak{m}$; i.e.,

$$R \to F^e_* R \xrightarrow{\cdot F^e_* c} F^e_* R$$

splits. □

Remark 1.13.20. If $\mathfrak{a}$ is a compatible ideal and $\tau(R) \cap \mathfrak{a} \neq 0$, then $\tau(R) \subseteq \mathfrak{a}$. It also makes sense to ask for the "largest" proper compatible ideal. This is easier to construct when R is noetherian by Zorn's lemma.

Definition 1.13.21 (Aberbach-Enescu ideals)**.** Let $(R, \mathfrak{m}, k)$ be a local ring with perfect residue field. For each e, define the **Aberbach-Enescu ideals**

$$\mathfrak{a}_e = \{r \in R \mid \varphi(F^e_* r) \in \mathfrak{m} \text{ for all } \varphi \in \operatorname{Hom}_R(F^e_* R, R)\}.$$

That is, these are the elements $r \in R$ for which

$$R \to F^e_* R \xrightarrow{\cdot F^e_* r} F^e_* R$$

does not split.

Definition 1.13.22 (F-splitting prime)**.** Define $\mathfrak{p}_s(R)$ to be

$$\mathfrak{p}_s(R) = \bigcap_e \mathfrak{a}_e = \left\{ r \in R \mid R \to F^e_* R \xrightarrow{\cdot F^e_* R} F^e_* R \text{ does not split for any } e \right\}.$$

One calls $\mathfrak{p}_s(R)$ the ***F*-splitting prime**.

Remark 1.13.23. Note that if R is not F-split, then $1 \in \mathfrak{p}_s(R)$; i.e., $\mathfrak{p}_s(R) = R$.

Theorem 1.13.24 (Aberbach-Enescu)**.** *If R is F-split, then $\mathfrak{p}_s(R)$ is prime.*

Proof. Since R is F-split, it is reduced. We can identify $F^e_* R \cong R^{\frac{1}{p^e}}$. Suppose that $ab \in \mathfrak{p}_s(R)$. Set

$$\varphi_{x,e} : R \to R^{\frac{1}{p^e}} \xrightarrow{\cdot x^{\frac{1}{p^e}}} R^{\frac{1}{p^e}}.$$

Proceed by contradiction and assume that $a \notin \mathfrak{p}_s(R)$ and $b \notin \mathfrak{p}_s(R)$. Choose e_1 and e_2 so that φ_{a,e_1} and φ_{b,e_2} split. Write $\psi_{x,e}$ for the splitting of $\varphi_{x,e}$. The "composition," up to identifying isomorphism classes, of φ_{a,e_1} and φ_{b,e_2}, will split. Let

$$\psi \in \operatorname{Hom}_R\left(R^{\frac{1}{p^{e_1+e_2}}}, R\right)$$

send $(a^{p^{e_2}} b)^{\frac{1}{p^{e_1+e_2}}}$ to 1. Ultimately, $\varphi_{a^{p^{e_2}} b, e_1+e_2}$ precomposed with ψ is a splitting. As $ab \in \mathfrak{p}_s(R)$, the element $a^{p^{e_2}} b$ is also in $\mathfrak{p}_s(R)$, which is a contradiction. □

Remark 1.13.25. We see that if R is F-split, $\mathfrak{p}_s(R) \neq R$ is prime. On the other hand, what happens when $\mathfrak{p}_s(R) = 0$?

Theorem 1.13.26. *Let R be an F-split ring. The following are equivalent:*

1. *R is F-regular,*
2. *$\tau(R) = R$, and*
3. *$\mathfrak{p}_s(R) = 0$.*

Proof. **Theorem 1.13.19** proves that 1 and 2 are equivalent. Note that $\mathfrak{p}_s(R) = 0$ if and only if for each $c \in R \setminus \{0\}$, there exists $e \gg 0$ such that

$$R \to F_*^e R \xrightarrow{\cdot F_*^e c} F_*^e R$$

splits. $\square$

Remark 1.13.27. For each e, we can set

$$\mathrm{im}_e = \mathrm{im}\left(\mathrm{Hom}_R(F_*^e R, R) \xrightarrow{ev_{F_*^e 1}} R\right),$$

and it's easy to check that

$$\cdots \subseteq \mathrm{im}_3 \subseteq \mathrm{im}_2 \subseteq \mathrm{im}_1 \subseteq R.$$

By a theorem of Hartshorne-Speiser-Lyubeznik, via Matlis dualilty, this chain stabilizes to an ideal which we denote $\sigma(R) = \mathrm{im}_e$ for any $e \gg 0$. This ideal is called the non-F-pure ideal. It's straightforward to check that $\sigma(R) = R$ if and only if R is F-split.

Definition 1.13.28 (pair)**.** Let R be a ring. Let M be an R-module. Let $\varphi \in \mathrm{Hom}_R(F_*^e M, M)$. We call the data (M, φ) a **pair**.

Example 1.13.29. Let R be regular. Let $\varphi = \Phi \in \mathrm{Hom}_R(F_* R, R)$ be a generator as an $F_* R$-module. The data (R, Φ) is a pair.

Example 1.13.30. The pair (ω_R, T) is the "dual" of (R, Φ).

Definition 1.13.31 (test ideal 2)**.** For any pair (M, φ), define the smallest nonzero φ-compatible submodule $\tau(M, \varphi)$. Call this, if it exists, the **test submodule** of (M, φ).

Example 1.13.32. Let $R = \mathbf{F}_2[x, y]$. As R is regular, $\tau(R) = R$. However, if we set $\{1, x^{\frac{1}{2}}, y^{\frac{1}{2}}, (xy)^{\frac{1}{2}}\}$ a basis for $R^{\frac{1}{2}}$ and define a map φ such that

$$\begin{aligned} 1 &\mapsto 0 \\ x^{\frac{1}{2}} &\mapsto 1 \\ y^{\frac{1}{2}} &\mapsto 0 \\ (xy)^{\frac{1}{2}} &\mapsto 0, \end{aligned}$$

then we claim $\tau(R, \varphi) = (y)$. Indeed, $\varphi(y^{\frac{1}{2}}) = 0 \in (y)$ and for any $f \in (y)$, there is a polynomial g so that $\varphi(f^{\frac{1}{2}}) = \varphi(yg^{\frac{1}{2}})$. We can see this by expanding $f^{\frac{1}{2}}$ and checking which terms go to 0. Thus,

$$\varphi\left(f^{\frac{1}{2}}\right) = \varphi\left(yg^{\frac{1}{2}}\right) = y\varphi\left(g^{\frac{1}{2}}\right) \in (y).$$

Thus (y) is φ-compatible. It's then a degree check to verify that $\tau(R, \varphi) = (y)$.

Additionally, we can compute the test ideals for the following pairs:

1. If $\varphi = \Phi$ is

$$\begin{aligned} 1 &\mapsto 0 \\ x^{\frac{1}{2}} &\mapsto 0 \\ y^{\frac{1}{2}} &\mapsto 0 \\ (xy)^{\frac{1}{2}} &\mapsto 1, \end{aligned}$$

then $\tau(R, \varphi) = R$.

2. If φ is

$$\begin{aligned} 1 &\mapsto 0 \\ x^{\frac{1}{2}} &\mapsto 0 \\ y^{\frac{1}{2}} &\mapsto 1 \\ (xy)^{\frac{1}{2}} &\mapsto 0, \end{aligned}$$

then $\tau(R,\varphi) = (x)$.

3. φ is

$$\begin{aligned} 1 &\mapsto 1 \\ x^{\frac{1}{2}} &\mapsto 0 \\ y^{\frac{1}{2}} &\mapsto 0 \\ (xy)^{\frac{1}{2}} &\mapsto 0, \end{aligned}$$

then $\tau(R,\varphi) = (xy)^{\frac{1}{2}?}$.

Remark 1.13.33. Additionally, finding any $c \in \tau(R,\varphi) \setminus \{0\}$, one has for $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$,

$$\tau(R,\varphi) = \sum_n \varphi^n(F^{ne}_* cR).$$

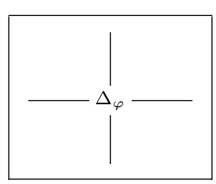
Lemma 1.13.34. *If $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ and $m \geq 1$, then $\tau(R,\varphi) = \tau(R,\varphi^m)$. Moreover, $\tau(R,\varphi^m) = \tau(R,\varphi^n)$ for all $m,n \geq 1$.*

Proof. It's clear that the second claim follows from the first, so we only show the first. It's also clear that $\tau(R,\varphi^m) \subseteq \tau(R,\varphi)$, since $\tau(R,\varphi)$ is φ^m-compatible. Conversely, pick c a test element for $\tau(R,\varphi^m)$ and write

$$\tau(R,\varphi) = \sum_n \varphi^n(F^{ne}_* cR) \subseteq \sum_n (\varphi^m)^n(F^{nme}_* cR) = \tau(R,\varphi^m).$$

□

Remark 1.13.35. We will see in semester two a geometric interpretation of pairs. In particular, set $X = \operatorname{Spec} R$. A "fancy" adjunction will associate to φ a subscheme $\Delta_\varphi \subseteq X$.



X

The ideal $\tau(R,\varphi)$ will characterize the singularities of Δ_φ as embedded in X.

Remark 1.13.36. Provisionally, one can define (R,φ) to be F-split if φ is a splitting of F^e and F-regular if $\tau(R,\varphi) = R$. It is elementary to check that for a domain R, (R,φ) is F-regular if and only if for all $c \in R \setminus \{0\}$, there exists $n \gg 0$ such that

$$R \to F^{ne}_* R \xrightarrow{\cdot F^{ne}_*} R$$

splits via the map $\varphi^n : F^{ne}_* R \to R$.

We will eventually see a greater generalization of pairs.

Remark 1.13.37. For Gorenstein rings, there is an obvious, nearly canonical, pair. Recall that $R \xrightarrow{F^e} F^e_* R$ dualizes to the trace map $F^e_*\omega_R \xrightarrow{T^e} \omega_R$. After identifying $\omega_R \cong R$ (which is not done canonically), we obtain

$$\begin{tikzcd} F^e_*\omega_R \arrow[r, "T^e"] \arrow[d, phantom, "\wr\|"] & \omega_R \arrow[d, phantom, "\wr\|"] \\ F^e_* R \arrow[r, "\Phi^e"'] & R \end{tikzcd}$$

Later, we will check that Φ^e generates $\operatorname{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module. (This is a consequence of adjunction.)

Theorem 1.13.38. *If R is a Gorenstein ring, then $\tau(R) = \tau(R, \Phi^e)$.*

Proof. It's clear that $\tau(R, \Phi^e) \subseteq \tau(R)$, as $\tau(R)$ is Φ^e-compatible.

Fix d, and write $\varphi \in \operatorname{Hom}_R(F^d_* R, R)$ as $\varphi(-) = \Phi^d(F^d_* c \cdot -)$ for some $c \in R$ by the generation claim in **Remark 1.13.37** above. We have

$$\varphi(F^d_*\tau(R, \Phi^e)) = \Phi^d(F^d_* c\tau(R, \Phi^e)) = \Phi^d(F^d_* c\tau(R, \Phi^d)),$$

by **Lemma 1.13.34**. Next,

$$\Phi^d(F^d_* c\tau(R, \Phi^d)) \subseteq \Phi^d(F^d_*\tau(R, \Phi^d)) \subseteq \tau(R, \Phi^d),$$

by compatibility. Finally,

$$\tau(R, \Phi^d) = \tau(R, \Phi^e)$$

by **Lemma 1.13.34**. Hence $\tau(R, \Phi^e)$ is φ-compatible for any φ. $\square$

Remark 1.13.39. Our next goal is to show that test ideals, in particular, $\tau(\omega_R, T)$, exist, for ω_R a canonical module and $T : F_*\omega_R \to \omega_R$ the dual of Frobenius. Recall that

$$\tau(\omega_R, T) = \sum_{e \geq 1} T^e(F^e_* c\omega_R)$$

for $c \in R \setminus \{0\}$.

Remark 1.13.40. For any domain, ω_R is finitely generated, torsion-free, and rank 1. That is,

$$\operatorname{rank}\omega_R = \dim_K(\omega_R \otimes_R K)$$

for $K = \operatorname{Frac} R$.

Theorem 1.13.41. *Let R be a domain. Let M be a noetherian, finitely generated, torsion-free, rank* 1 *R-module. Let $\varphi \neq 0$ so that (M, φ) is a pair. The test ideal $\tau(M, \varphi)$ exists.*

Proof. As M has rank 1, it is possible to find $c \neq 0$ such that $M_c = M \otimes_R R_c \cong R_c$, and for any fixed e,
1. $F^e_* M_c \cong F^e_* R_c$,
2. $cM \subseteq \varphi(F^e_* M)$, and
3. the map $\varphi_c : F^e_* M_c \to M_c$ generates $\operatorname{Hom}_{R_c}(F^e_* M_c, M_c)$ as an $F^e_* R_c$-module.

Note: finding such a c in practice is fairly easy. To see how these can be satisfied, note that φ is "generically surjective" (i.e., φ_c is surjective for all $c \in R \setminus V(\mathfrak{a})$ for some ideal $\mathfrak{a}$), and $\operatorname{Hom}_R(F^e_* M, M)$ also has rank 1.

> For example, set $K = \operatorname{Frac} R$. As M has rank 1, $M \otimes_R K \cong K$. After clearing denominators, any $\pi \otimes 1$ has $c\pi = 0$ whenever $\pi \in M$ is torsion. That is, if $c \in \operatorname{Ann} \pi$, then
>
> $$\pi \otimes 1 = \pi \otimes \frac{c}{c} = c\pi \otimes \frac{1}{c} = 0 \otimes \frac{1}{c} = 0.$$
>
> As M is finitely generated and noetherian, the torsion submodule of M is finitely generated, so working over all generators of the torsion submodule, we can find a single c such that $c\pi = 0$ for all torsion π. Thus, in M_c, $\pi = 0$, so $M_c \cong R_c$, and thus $F^e_* M_c \cong F^e_* R_c$. So property 1 is believable.

Next, we make the following claim:

Claim. *For any $N \subseteq M$ that is φ-compatible, $N_c \cong M_c \cong R_c$.*

Proof. To establish the claim, it suffices to prove that $M_{\mathfrak{q}} \cong N_{\mathfrak{q}} \cong R_{\mathfrak{q}}$ for all $\mathfrak{q} \in \operatorname{Spec} R_c$. Pick $n \in N_{\mathfrak{q}} \setminus \{0\}$ and $\ell \gg 0$ such that $F_*^{\ell e} n \notin \mathfrak{q} F_*^{\ell e} M_{\mathfrak{q}}$. Otherwise, $F_*^{\ell e} N_{\mathfrak{q}} \subseteq \mathfrak{q} F_*^{\ell e} M_{\mathfrak{q}}$ implies $N_{\mathfrak{q}} \cong M_{\mathfrak{q}}$, and we are done.

Now, $F_*^{\ell e} n \notin \mathfrak{q} F_*^{\ell e} M_{\mathfrak{q}} \cong F_*^{\ell e}\left(\mathfrak{q}^{[p^{\ell e}]} R_{\mathfrak{q}}\right)$, This is because $M_c \cong R_c$, which implies $M_{\mathfrak{q}} \cong R_{\mathfrak{q}}$ for all $\mathfrak{q} \in \operatorname{Spec} R_c$.

Also, $F_*^{\ell e} M_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module, so $F_*^{\ell e}\left(M_{\mathfrak{q}} / \mathfrak{q}^{[p^{\ell e}]}\right)$ is a free $R_{\mathfrak{q}} / \mathfrak{q}$-module of the same rank. Choose $\overline{a_1} = n$ and $\overline{a_2}, ..., \overline{a_s}$ as a basis for $F_*^{\ell e}\left(M_{\mathfrak{q}} / \mathfrak{q}^{[p^{\ell e}]}\right)$ as an $R_{\mathfrak{q}} / \mathfrak{q}$-module. This produces a map

$$\gamma : \bigoplus_i a_i R \to F_*^{\ell e} M_{\mathfrak{q}}$$

which is surjective by **Lemma 1.2.9 [Nakayama's Lemma]**. By rank consideration, γ is bijective. (That is, a surjective map of free modules of the same rank is bijective.) Projection onto the first coordinate defines a map $\psi : F_*^{\ell e} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ such that $\psi(F_*^{\ell e} n R_{\mathfrak{q}}) = M_{\mathfrak{q}}$. By property 3, we can write ψ as $\varphi^{\ell}(F_*^{\ell e} d \cdot -)$, and therefore

$$M_{\mathfrak{q}} = \psi(F_*^{\ell e} n R_{\mathfrak{q}}) \subseteq \psi(F_*^{\ell e} N_{\mathfrak{q}}) = \varphi^{\ell}(F_*^{\ell e} d N_{\mathfrak{q}}) \subseteq \varphi^{\ell}(F_*^{\ell e} N_{\mathfrak{q}}) \subseteq N_{\mathfrak{q}} \subseteq M_{\mathfrak{q}},$$

Therefore, $M_{\mathfrak{q}} \cong N_{\mathfrak{q}}$, as claimed. □

Let's see how this claim proves the theorem. Since $N_c \cong M_c$, $c^n M \subseteq N$, as M is finitely generated. In particular, for $m \in M$,

$$\frac{m}{1} = \frac{\eta}{c^n}$$

for some $\eta \in N$ and $n \in \mathbf{W}$, so $c^n m = \eta \in N$. Working over a finite generating set, we can pick n. (In fact $n = 2$ works!) Set $t \gg 0$ so that $p^{et} \geq n + 1$. We have

$$c^2 M \subseteq ccM \subseteq c\varphi^t(F_*^{te} M) = \varphi^t(F_*^{te} c^{p^{te}} M) \subseteq \varphi^t(F_*^{te} c^n M) \subseteq \varphi^t(F_*^{te} N) \subseteq N,$$

by property 2, the fact that $p^{et} \geq n + 1$, and the fact that N is φ-compatible.

Finally,

$$\sum_t \varphi^t(F_*^{te} c^2 M) \subseteq N,$$

but the left side is φ-compatible. As N is arbitrary,

$$\tau(M, \varphi) = \sum_t \varphi^t(F_*^{te} c^2 M),$$

as desired. □

Corollary 1.13.42. *If R is Gorenstein, then $\tau(R) = \tau(\omega_R, \Phi)$ exists.*

1.13.1 Tight Closure

Remark 1.13.43. We have not shown that $\tau(R)$ exists in full generality. The existence of $\tau(R)$ depends on tight closure.

Remark 1.13.44. Tight closure will be able to give us the following applications:

1. **Remark 1.8.44**: A consequence of the Briançon-Skoda Theorem is that for a local ring $(R, \mathfrak{m}, k)$ of dimension d with k infinite and R F-rational, if J is an ideal such that $\mathfrak{m}^n = J\mathfrak{m}^{n-1}$ (we say J is a reduction of $\mathfrak{m}$), then $\mathfrak{m}^d \subseteq J$. Recall that we used this to show **Theorem 1.8.41 [Huneke-Watanabe]**, which said
$$e(R) = \lim_{n\to\infty} \frac{d!\lambda\left(R/\mathfrak{m}^n\right)}{n^d} \leq \binom{\nu-1}{d-1},$$
where ν is the embedding dimension of R.
2. **Theorem 1.4.50 [Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede]**: If R is a regular ring, $\mathfrak{a} \subseteq R$ is an ideal with bight $\mathfrak{a} = h$, then $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$ for all n.
3. Let R be a domain of finite type over k. Let $\mathfrak{a} = (f_1, ..., f_h)$ be a complete intersection in R; i.e., $\operatorname{ht}\mathfrak{a} = h$. There exists an ideal J such that $JH^i_{\mathfrak{a}}(R) = 0$ for all $i < h$. J is the Jacobian. Explicitly, if $R = {k[x_1, ..., x_d]}/{(g_1, ..., g_t)}$, then
$$J = \operatorname{Jac} R = I_r \left[\frac{\partial g_i}{\partial x_j}\right],$$
that is, the $r \times r$ minors of the matrix of partial derivatives, where $r = d - \dim R$.
4. Tight closure will give meaning to $0^*_{H^d_{\mathfrak{m}}(R)}$.
5. Tight closure will give new proofs of
$$\begin{array}{ccc} F\text{-regular} & \Longrightarrow & F\text{-rational} \\ \Downarrow & & \Downarrow \\ F\text{-split} & \Longrightarrow & F\text{-injective} \end{array}$$
6. Tight closure will relate to questions about splinters.

Definition 1.13.45 (integral element)**.** For a ring extension $R \subseteq S$, an element $t \in S$ is **integral** over R provided that t is the root of a monic polymonial $f \in R[x]$.

Definition 1.13.46 (integral extension)**.** An extension $R \subseteq S$ is **integral** if every element in S is integral over R.

Example 1.13.47. Let $K \subseteq L$ be a field extension. The algebraic elements of L over K are integral over K.

Example 1.13.48. The extension $\mathbf{Z} \subseteq \mathbf{Z}[\sqrt{d}]$ where d is a positive integer is an integral extension, since $\sqrt{d}$ is the root of $x^2 - d \in \mathbf{Z}[x]$.

Example 1.13.49. The element $\frac{1+\sqrt{5}}{2} \in \mathbf{Q}[\sqrt{5}]$ is integral over $\mathbf{Z}[\sqrt{5}]$ using $T^2 - T - 1$.

Example 1.13.50. Let $R = k[y]$ and $S = {k[y,t]}/{(t^2-y)}$. The element $t \in S$ is integral over R via $x^2 - y$. Using toric rings, this extension is equivalent to $k[t^2] \subseteq k[t]$.

Example 1.13.51. Let $R = k[x]$ and $S = {k[x,y]}/{(x^2+y^3)}$. The element $y \in S$ is integral over R via $X^3 + x^2$. This extension is equivalent to $k[t^3] \subseteq k[t^2, t^3]$.

Remark 1.13.52. Recall **Example 1.4.12**. Let G be a group. An action of G on the ring we denote $R = k[x_1, ..., x_n]$ is an embedding $G \to \operatorname{Aut}_k(R)$, where $\operatorname{Aut}_k(R)$ is the set of k-linear automorphisms of R. This defines a subring $R^G = \{f \in R \mid gf = f \text{ for all } g \in G\} \subseteq R$. If $|G| < \infty$ and $\gcd(|G|, \operatorname{char} k) = 1$, then the Reynolds operator $\rho : R \to R^G$ sending

$$\rho(f) = \frac{1}{|G|} \sum_{g \in G} gf$$

is a splitting of $R^G \hookrightarrow R$; i.e., $R \cong R^G \oplus L$.

Lemma 1.13.53. *Let R be a direct summand of S.*

1. *For any ideal $\mathfrak{a} \subseteq R$, $\mathfrak{a}S \cap R = \mathfrak{a}$.*
2. *If S is noetherian, then so is R.*

Proof.

1. The inclusion $\mathfrak{a} \subseteq \mathfrak{a}S \cap R$ is clear. To see $\mathfrak{a}S \cap R \subseteq \mathfrak{a}$, choose $x \in \mathfrak{a}S \cap R$ and write $x = \sum a_i s_i$. Now note that $\rho(x) = \sum a_i \rho(s_i)$ by linearity, where $\rho : S \to R$ is the splitting map. Send the image back along the inverse of the splitting to see $x \in \mathfrak{a}$.
2. Obvious.

□

Theorem 1.13.54 (Hochster)**.** *Let $R \subseteq S$ be a module finite extension with R a reduced excellent ring. R is a direct summand of S if and only if $\mathfrak{a}S \cap R = \mathfrak{a}$ for all $\mathfrak{a} \subseteq R$.*

Remark 1.13.55. There is a connection with R^G to Hilbert's fourteenth problem. If we have $R^G \hookrightarrow R = k[x_1, ..., x_n]$ and R^G is noetherian, when the action is degree preserving, then R^G is a finitely generated k-algebra.

Theorem 1.13.56. *Let $A \subseteq B \subseteq C$ be ring extensions. If A is noetherian, C is a finitely generated A-algebra, and C is a finitely generated B-module, then B, as an A-algebra, is also finitely generated.*

Theorem 1.13.57. *Let $R \subseteq S$ be a ring extension. The following are equivalent:*

1. *S is a finitely generated R-module, and*
2. *S is a finitely generated R-algebra and S is integral over R.*

Corollary 1.13.58. *Let $R \subseteq S$ be an extension of rings. The set of elements of S which are integral over R forms a subring of S.*

Proof. It suffices to show that any sum or product of integral elements is integral. (Note that strictly from the definition, the product is okay, but the sum would be hard.) Let $s_1, s_2 \in S$ be integral over R. Note that s_2 is integral over $R[s_1]$, and consider the diagram

$$\begin{tikzcd} R[s_1, s_2] \\ R[s_1] \arrow[u, dash, "\text{integral, module finite}"'] \\ R \arrow[u, dash, "\text{integral, module finite}"'] \end{tikzcd}$$

Therefore, $R \subseteq R[s_1, s_2]$ is a module finite extension. By **Theorem 1.13.57**, $R[s_1, s_2]$ is integral over R, as we wished to show. □

Theorem 1.13.59 (Noether)**.** *Let G be a finite group acting in a degree preserving manner on $R = k[x_1, ..., x_n]$. The ring of invariants R^G is a finitely generated k-algebra.*

Proof. We will apply **Theorem 1.13.56** to the extensions $k \subseteq R^G \subseteq R$. To check its hypotheses, see first that k is a field, hence noetherian. See second that R is a finitely generated k-algebra. To see third that R is a finitely generated R^G-module, we use **Theorem 1.13.57**. First, R is a finitely generated R^G-algebra. We just need to show that $R^G \to R$ is integral. It suffices to show $x_i \in R$ is integral over R^G for $i \in \{1, ..., n\}$. Note that x_i is a root of the polynomial

$$\prod_{g \in G} (T - gx_i).$$

The coefficients of this polynomial are in R^G. The result follows. $\square$

Definition 1.13.60 (total ring of fractions)**.** Given any ring R, we can construct

$$K(R) = \left\{ \frac{a}{b} \mid a, b \in R, b \text{ is not a zero divisor} \right\},$$

the **total ring of fractions** of R.

Remark 1.13.61. When R is a domain, $K(R) = \operatorname{Frac} R$ is a field.

Example 1.13.62. $K(\mathbf{Z}) = \mathbf{Q}$.

Example 1.13.63. $K(k[x_1, ..., x_n]) = k(x_1, ..., x_n)$.

Definition 1.13.64 (normalization)**.** The **normalization** of R is the ring

$$R^N = \{z \in K(R) \mid z \text{ is integral over } R\}.$$

Definition 1.13.65 (normal)**.** R is **normal** if $R = R^N$.

Remark 1.13.66. One can show that $(W^{-1}R)^N \cong W^{-1}(R^N)$ for any W a multiplicatively closed set. In particular, if R is normal, then $R_\mathfrak{p}$ is normal for all $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 1.13.67. Recall that a ring R is reduced if 0 is the only nilpotent element in R. That is, if $x^n = 0$, then $x = 0$.

Example 1.13.68. If $R = k[x_1, ..., x_d]/\mathfrak{a}$ where $\mathfrak{a}$ is a squarefree monomial ideal, then R is reduced. Recall that an ideal like $\mathfrak{a} = (xy, xzw)$ is squarefree, while $\mathfrak{a} = (x^2y, xzw)$ is not.

Lemma 1.13.69. *If $(R, \mathfrak{m})$ is a local ring that is reduced and normal, then R is a domain.*

Proof. Let $\{\mathfrak{q}_1, ..., \mathfrak{q}_r\}$ be the minimal primes of R. One can check that if R is reduced, then $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. Hence,

$$\begin{aligned} R &\hookrightarrow \prod_{i=1}^r R/\mathfrak{q}_i \\ a &\mapsto (\overline{a}, ..., \overline{a}). \end{aligned}$$

One can check that

$$K(R) = \prod_{i=1}^r K\left(R/\mathfrak{q}_i\right).$$

Note that $\prod R/\mathfrak{q}_i$ is a finitely generated R-module, so its elements are integral over R by **Theorem 1.13.57**. Thus $\prod R/\mathfrak{q}_i \subseteq K(R)$. Since R is normal,

$$R = \prod_{i=1}^r R/\mathfrak{q}_i.$$

Since R is local, $r = 1$. $\square$

Definition 1.13.70 (Serre's condition (R_k))**.** We say that R has **(Serre's condition)** (R_k) if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$ of height at most k. (In other words, R is regular in codimension k.)

Definition 1.13.71 (Serre's condition (S_k))**.** We say that R has **(Serre's condition)** (S_k) if $\operatorname{depth} R_{\mathfrak{p}} \geq \min\{k, \dim R_{\mathfrak{p}}\}$. (In other words, R is Cohen-Macaulay in codimension k.)

Remark 1.13.72. Serre's conditions are useful because they are purely homological conditions. It is therefore valuable to show that nonhomological conditions are equivalent to Serre's conditions.

Example 1.13.73. R is regular if and only if R has (R_k) for all k.

Example 1.13.74. If $X = \operatorname{Spec} R$ is a surface, then R has (R_1) if and only if X has only isolated singularities.

Example 1.13.75. Let $R = {}^{k[x,y,z,w]}\!\big/_{(xz,xw,yz,yw)}$. The ring R has (R_1) but not (R_2). Since $(xz, xw, yz, yw) = (x, y) \cap (z, w)$, we are unioning two planes in $\mathbf{A}^4_k$, which is only nonsingular in the origin.

Lemma 1.13.76. *If R is a UFD, then R is a normal domain.*

Proof. Let $z \in K(R)$ be integral over R. Write $z = \frac{a}{b}$ with $\gcd(a, b)$ a unit in R. Since z is integral, there exists $n \in \mathbf{N}$ and $a_0, ..., a_{n-1} \in R$ such that

$$\left(\frac{a}{b}\right)^n + a_{n-1}\left(\frac{a}{b}\right)^{n-1} + \cdots + a_0 = 0.$$

Since R is a domain, multiply by b^n to get

$$a^n + ba_{n-1}a^{n-1} + \cdots + b^n a_0 = 0.$$

Thus $a^n \in (b)$. However, since $\gcd(a, b) = u$, $\gcd(a^n, b) = u'$, and since b divides a^n, $\gcd(a^n, b) = b$. Thus, b is a unit. Therefore $z = \frac{a}{b} = ab^{-1} \in R$. $\square$

Remark 1.13.77. Under mild conditions, an R-module M has (S_2) if and only if M is reflexive; i.e.,

$$M \cong M^{\vee\vee} = \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R).$$

Theorem 1.13.78. *A ring R is normal if and only if R has (R_1) and (S_2).*

Proof. We show only that if R is normal, then R is (R_1). Let R be normal. To show R is (R_1), we need to show that 1-dimensional normal local rings are regular, since $\dim R_{\mathfrak{q}} = \operatorname{ht} \mathfrak{q}$. It's enough to show that $\mathfrak{m}$ is principal. Take $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, assume for the sake of contradiction that there exists $y \in \mathfrak{m} \setminus (x)$. Let $\mathfrak{m}^{-1} = R :_{K(R)} \mathfrak{m} = \{z \in K(R) \mid z\mathfrak{m} \subseteq R\}$.

We claim that $\mathfrak{m}\mathfrak{m}^{-1} = R$. Note that $\mathfrak{m} \subseteq \mathfrak{m}\mathfrak{m}^{-1}$, because for all $a \in \mathfrak{m}$, $a = a \cdot 1$. Note also that $\mathfrak{m}\mathfrak{m}^{-1} \subseteq R$. By maximality, we show that $\mathfrak{m}\mathfrak{m}^{-1} \neq \mathfrak{m}$. By contradiction, if $\mathfrak{m} = \mathfrak{m}\mathfrak{m}^{-1}$, then $y\mathfrak{m} \subseteq (x)$. Since $y \notin (x)$, $\frac{y}{x} \notin R$. However, $\frac{y}{x} \in \mathfrak{m}^{-1}$, so $\frac{y}{x}\mathfrak{m} \subseteq \mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$. By the determinental trick, there exists a monic equation for $\frac{y}{x}$ with coefficients in R. Since R is normal, $\frac{y}{x} \in R$, a contradiction. $\square$

Example 1.13.79. The ring $R = {}^{k[x,y,z,w]}\!\big/_{(xz,xw,yz,yw)}$ is not normal, since it does not have (S_2); $p\dim R = 3$, so $\operatorname{depth} R = 1$ by **Theorem 1.5.20 [Auslander-Buchsbaum]**. Yet $\dim R = 2$, so R does not have (S_2).

Remark 1.13.80. Recall that R is unmixed if $\operatorname{ht} \mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$. One can show that R is unmixed if and only if R has (S_1). Furthermore, one can show that R is reduced if and only if R has (R_0) and (S_1).

Remark 1.13.81. Returning to F-singularities, what is the connection between integral extensions and closures, normality, and Frobenius splittings/singularities?

Remark 1.13.82. For a ring R, there is a natural map $R \hookrightarrow R^N$ which extends to a short exact sequence

$$0 \to R \to R^N \to {}^{R^N}\!\diagup_{R} \to 0.$$

R is normal if and only if ${}^{R^N}\!\diagup_{R} = 0$. Consider the ideal

$$\mathfrak{c} = \operatorname{Ann}\left({}^{R^N}\!\diagup_{R}\right) = \{r \in R \mid rR^N \subseteq R\},$$

which we call the conductor ideal. It's easy to see that $\mathfrak{c}$ is an ideal of both R and R^N. In fact, $\mathfrak{c}$ is the largest simultaneous ideal of R^N and R; for any ideal $\mathfrak{a} \subseteq R^N$ which is also an ideal of R such that $\mathfrak{a}R^N \subseteq R$, one has $\mathfrak{a} \subseteq \mathfrak{c}$.

Theorem 1.13.83. *If R is an F-regular ring, then R is normal.*

Proof. Let $\mathfrak{c}$ be the conductor. We claim that $\mathfrak{c}$ is φ-compatible for φ-compatible for all maps $\varphi \in \operatorname{Hom}(F^e_* R, R)$. That is, $\tau(R) \subseteq \mathfrak{c}$, but since R is F-regular, $\mathfrak{c} \subseteq R = \tau(R) \subseteq \mathfrak{c}$, so ${}^{R^N}\!\diagup_{R} = 0$. Set $K(R)$ to be the total ring of quotients. For any $\varphi : F^e_* R \to R$, we may tensor by $K(R)$ to get $\psi : F^e_* K(R) \to K(R)$. Identify φ with $\psi|_R$. For any $x \in \mathfrak{c}$ and $r \in R^N$, we have

$$\varphi(F^e_* x)r = \psi(F^e_* r^{p^e} x)$$

with $r^{p^e} \in R^N$. Since $x \in \mathfrak{c}$, $r^{p^e} x \in R$, and thus

$$\varphi(F^e_* x)r = \psi(F^e_* r^{p^e} x) \in R.$$

Since $r \in R^N$ was arbitrary, we see $\varphi(F^e_* x)r \in R$, so $\varphi(F^e_* x) \in \mathfrak{c}$. That is, $\mathfrak{c}$ is φ-compatible. ☐

Definition 1.13.84 (complement of minimal primes)**.** Set, for any ring R, the set R°, which is the complement of the minimal primes of R. That is,

$$R^\circ = R \setminus \bigcup_{\mathfrak{p} \text{ a minimal prime}} \mathfrak{p}.$$

Remark 1.13.85. If R is a domain, then $R^\circ = R \setminus \{0\}$.

Remark 1.13.86. We could have defined F-regular for non-domains. For example, a ring R is F-regular if for any $c \in R^\circ$, the map $R \to F^e_* R$ given by $1 \mapsto F^e_* c$ splits for $e \gg 0$. Using the same proof as **Theorem 1.13.83** above, one can show that if R is a reduced F-regular ring, then R is normal, hence a domain by **Lemma 1.13.69**.

Example 1.13.87. It's easy to see examples of F-split rings that are not normal. The ring $R = {}^{k[x,y]}\!\diagup_{(xy)}$ is F-split using **Corollary 1.4.24 [Fedder's Criterion]**, but not regular/does not have (R_1), so by **Theorem 1.13.78**, R is not normal.

Definition 1.13.88 (seminormal)**.** For an integral extension of reduced rings $R \subseteq S$, call R **seminormal in** S provided for each pair of relatively prime integers c and d, if $b \in S$ and $b^c, b^d \in R$, then $b \in R$. We call a reduced ring R **seminormal** if it is seminormal in $R \hookrightarrow R^N$.

Definition 1.13.89 (weakly normal)**.** For an integral extension of reduced rings $R \subseteq S$ each of characteristic $p > 0$, call R **weakly normal in** S if for all $b \in S$, $b^p \in R$ implies $b \in R$. We call a reduced ring R weakly normal if it is weakly normal in $R \hookrightarrow R^N$.

Remark 1.13.90. In general, if R is normal, then it is weakly normal. If R is weakly normal, then it is seminormal.

Example 1.13.91. The ring $R = {k[x,y]}/{(xy)}$ is weakly normal, hence seminormal.

Example 1.13.92. The ring $R = {k[x,y]}/{(xy(x-y))}$ is not seminormal.

Example 1.13.93. The ring $R = {k[x,y,z]}/{(x^2y-z^2)}$ is seminormal. R is weakly normal if $p \neq 2$. Note that $\left(\frac{z}{x}\right)^2 = y^2 \in R$, but $\frac{z}{x} \notin R$.

Theorem 1.13.94. *If R is an F-split ring, then R is weakly normal.*

Proof. As R is F-split, let $\varphi : F_*R \to R$ be the splitting. Let $r \in R^N$ be such that $r^p \in R$. Apply φ to see that $r = r\varphi(F_*1) = \varphi(F_*r^p) \in R$. ☐

Definition 1.13.95 (integral closure)**.** Given an ideal $\mathfrak{a} \subseteq R$, we define the **integral closure** of $\mathfrak{a}$ to be the ideal

$$\begin{aligned}\overline{\mathfrak{a}} &= \{r \in R \mid r \text{ is integral over } \mathfrak{a}\}\\ &= \{r \in R \mid \text{there exists } c \neq 0 \text{ such that } cz^k \in \mathfrak{a} \text{ for all } k \gg 0\}.\end{aligned}$$

Remark 1.13.96. The integral closure $\mathfrak{a}$ of an ideal $\mathfrak{a}$ is a closure; that is, it satisfies the following:

1. $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$,
2. $\overline{\overline{\mathfrak{a}}} = \overline{\mathfrak{a}}$, and
3. if $\mathfrak{a} \subseteq \mathfrak{b}$, then $\overline{\mathfrak{a}} \subseteq \overline{\mathfrak{b}}$.

We know of other important closures in algebra. Given an ideal $\mathfrak{a}$, $\sqrt{\mathfrak{a}}$ is a closure. For a fixed $\mathfrak{m} \in \operatorname{Spec} R$,

$$\mathfrak{a}^{sat} = (\mathfrak{a} : \mathfrak{m}^\infty) = \bigcup_n (\mathfrak{a} : \mathfrak{m}^n) = \{r \in R \mid \mathfrak{m}^n r \in \mathfrak{a} \text{ for some power } n\}$$

is a closure. (Note that definitionally, $H^0_{\mathfrak{m}}\left({R}/{\mathfrak{a}}\right) = {\mathfrak{a}^{sat}}/{\mathfrak{a}}$.)

Remark 1.13.97. Working in characteristic $p > 0$, we have Frobenius powers $q = p^e$ with $\mathfrak{a}^{[q]} \subsetneq \mathfrak{a}^q$, so we can define the following analog of integral closure. Note that q will subsequently refer to a power of p; i.e., $q = p^e$, unless otherwise mentioned.

Definition 1.13.98 (tight closure)**.** Let R be any ring of characteristic $p > 0$. For any ideal $\mathfrak{a} \subseteq R$, set

$$\mathfrak{a}^* = \{z \in R \mid \text{there exists } c \in R^\circ \text{ such that } cz^q \in \mathfrak{a}^{[q]} \text{ for all } q \gg 0\}.$$

Call $\mathfrak{a}^*$ the **tight closure** of $\mathfrak{a}$. One can easily check that $\mathfrak{a}^*$ is an ideal.

Remark 1.13.99. The element c can depend on $\mathfrak{a}$ and z, but c does not depend on q.

Remark 1.13.100. The motivation for tight closure comes from the following. Let R be a reduced ring. Let $z \in \mathfrak{a}^*$ for some ideal $\mathfrak{a}$. Write $\mathfrak{a} = (f_1, ..., f_s)$, so that $\mathfrak{a}^{[q]} = ({f_1}^q, ..., {f_s}^q)$. We may then write

$$cz^q = \sum_i g_i {f_i}^q.$$

Viewing in $R \subseteq R^{\frac{1}{q}}$, we have

$$c^{\frac{1}{q}} z = \sum_i {g_i}^{\frac{1}{q}} f_i.$$

Roughly, as $q \to \infty$, $c^{\frac{1}{q}} z$ and ${g_i}^{\frac{1}{q}}$ approach 1. (The precise justification for this uses valuations.) Thus, z is "almost" in $\mathfrak{a}$.

Lemma 1.13.101. *The operation $\mathfrak{a} \mapsto \mathfrak{a}^*$ is a closure operator; i.e.,*

1. $\mathfrak{a} \subseteq \mathfrak{a}^*$
2. $\mathfrak{a}^{**} = \mathfrak{a}^*$, *and*
3. *if* $\mathfrak{a} \subseteq \mathfrak{b}$, *then* $\mathfrak{a}^* \subseteq \mathfrak{b}^*$.

Proof.

1. It's easy to verify $\mathfrak{a}^{[q]} \subseteq \mathfrak{a} \subseteq \mathfrak{a}^* \subseteq \overline{\mathfrak{a}}$.
2. Assume R is noetherian, so that ideals are finitely generated. First, if $\mathfrak{a}^* = (f_1, ..., f_s)$, then pick c_i such that $c_i {f_i}^q \in \mathfrak{a}^{[q]}$ for $q \gg 0$ and all i. Set $c = c_1 \cdots c_s$. Notice that $c{f_i}^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$ and all i. That is, $c(\mathfrak{a}^*)^{[q]} \subseteq \mathfrak{a}^{[q]}$.
   Now, if $z \in \mathfrak{a}^{**}$, pick c' such that $c'z^q \in (\mathfrak{a}^*)^{[q]}$ for $q \gg 0$. Multiply by c to see that $cc'z^q \in c(\mathfrak{a}^*)^{[q]} \subseteq \mathfrak{a}^{[q]}$ for $q \gg 0$. Therefore, $z \in \mathfrak{a}^*$, as desired.
3. It's easy to verify if $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{a}^* \subseteq \mathfrak{b}^*$.

$\square$

Example 1.13.102. Let $R = k[x^2, x^3] \subseteq k[x]$. Note that $x^3 \notin (x^2)$ in R. We claim that $x^3 \in (x^2)^*$. Indeed, see that

$$x^{3q} = x^q(x^{2q}) \in (x^2)^{[q]}$$

for $q \gg 0$. That is, here $c = 1$.

Example 1.13.103. Let $R = {}^{k[x,y,z]}\!/\!_{(x^3+y^3-z^3)}$ for $p \neq 3$. We claim that $(x, y)^* = (x, y, z^2)$. We will just show that $z^2 \in (x, y)^*$. Write $p \equiv r \bmod 3$. We have

$$(z^2)^q = z^{2q} = z^{2q-r}z^r = (z^3)^{\frac{2q-r}{3}} z^r = (x^3 + y^3)^{\frac{2q-r}{3}} z^r = z^r \cdot \sum \binom{\frac{2q-r}{3}}{i} x^{3i} y^{3\left(\frac{2q-r}{3} - i\right)}.$$

One can check that each $x^m y^n$ has $m \geq q$ or $n \geq q$, unless $r = 1$ and $m = n = q - 1$. That is, $(z^2)^q \notin (x^q, y^q)$. However, if we set $c = x$ (or y), then $c(z^2)^q \in (x^q, y^q)$; i.e., $z^2 \in (x, y)^*$.

Definition 1.13.104 (weakly F-regular)**.** Call a ring R **weakly F-regular** if $\mathfrak{a}^* = \mathfrak{a}$ for all $\mathfrak{a} \subseteq R$.

Remark 1.13.105. It is not known that if R is a weakly F-regular ring, then for all multiplicatively closed sets $W \subseteq R$, $W^{-1}R$ is weakly F-regular.

Definition 1.13.106 (F-regular)**.** A ring R for which $W^{-1}R$ is weakly F-regular for all multiplicatively closed subsets $W \subseteq R$ is called **F-regular**.

Remark 1.13.107. Recall that all prior mentions of F-regular rings were referring to strongly F-regular rings. Recall **Definition 1.9.2 [stongly F-regular]**. For each $c \in R^\circ$, there exists $e \gg 0$ such that there is a $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ with $\varphi(F^e_* c) = 1$. Recall also that **Theorem 1.13.19** characterizes strongly F-regular rings as those for which $\tau(R) = R$.

Remark 1.13.108. In general, if R is a strongly F-regular ring, then R is F-regular. If R is an F-regular ring, then R is weakly F-regular. There is a conjecture [weak = strong] that if R is a weakly F-regular ring, then R is strongly F-regular. It is still open, but it is known to be true for $\mathbf{N}$-graded rings, Gorenstein rings, rings of invariants of "nice" groups, determinental rings, and others.

Lemma 1.13.109. *If R is strongly F-regular, then R is F-regular.*

Proof. Note that both conditions are local, so it suffices to assume that R is a local domain. Fix $\mathfrak{a} \subseteq R$. Suppose that $z \in \mathfrak{a}^*$; i.e., there exists $c \neq 0$ such that $cz^q \in \mathfrak{a}^{[q]}$. Write $\mathfrak{a} = (f_1, ..., f_s)$; then we have $\mathfrak{a}^{[q]} = ({f_1}^q, ..., {f_s}^q)$. Pick $q \gg 0$ with $\varphi : R^{\frac{1}{q}} \to R$ that sends $c^{\frac{1}{q}}$ to 1. Thus

$$cz^q = \sum g_i {f_i}^q,$$

so

$$c^{\frac{1}{q}}z = \sum g_i{}^{\frac{1}{q}} f_i.$$

Applying φ, we see that

$$z = \varphi\left(c^{\frac{1}{q}}z\right) = \varphi\left(\sum g_i{}^{\frac{1}{q}} f_i\right) = \sum \varphi\left(g_i{}^{\frac{1}{q}}\right) f_i \in \mathfrak{a}.$$

□

Remark 1.13.110. Recalling **Example 1.9.4**, we know that regular rings are strongly F-regular. We thus have the implications regular implies strongly F-regular implies weakly F-regular. (Hence, weakly F-regular is a singularity type.) However, historically, the notion of weakly F-regular came first, so let's see a classic proof of the following:

Theorem 1.13.111 (Hochster-Huneke)**.** *A regular ring is weakly F-regular.*

Proof. Let R be regular. By **Theorem 1.1.24 [Kunz]**, the Frobenius is flat, so for any z, any $\mathfrak{a}$, and $q \gg 0$,

$$\left(\mathfrak{a}^{[q]} : z^q\right) = (\mathfrak{a} : z)^{[q]}.$$

(Indeed, to check this, take the short exact sequence

$$0 \to R/(\mathfrak{a} : z) \xrightarrow{\cdot z} R/\mathfrak{a} \to R/(\mathfrak{a} + zR) \to 0$$

and tensor by $F^e_* R$.)
If $z \in \mathfrak{a}^*$, then $cz^q \in \mathfrak{a}^{[q]}$, so $c \in (\mathfrak{a}^{[q]} : Z^q) = (\mathfrak{a} : z)^{[q]}$, and $c \neq 0$. This forces

$$c \in \bigcap_{n \in \mathbf{N}} (\mathfrak{a} : z)^n$$

by the cofinality of ordinary powers and Frobenius powers. Since $c \neq 0$, this means $(\mathfrak{a} : z) = R$; i.e., $z \in \mathfrak{a}$. □

Theorem 1.13.112 (Hochster-Huneke)**.** *If R is any ring and $\mathfrak{a} = (f_1, ..., f_n) \subseteq R$ is an ideal, then $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}^*$.*

Proof. Suppose $z \in \overline{\mathfrak{a}^n}$; i.e., there exists $c \neq 0$ such that $cz^m \in (\mathfrak{a}^n)^m$. Apply this with $m = p^e = q$ to see that $cz^q \in \mathfrak{a}^{nq} \subseteq \mathfrak{a}^{[q]}$, so $z \in \mathfrak{a}^*$. □

Corollary 1.13.113 (Brianąn-Skoda)**.** *If R is weakly F-regular and $\mathfrak{a} = (f_1, ..., f_n)$, then $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$.*

Remark 1.13.114. Using reduction mod p, one can use this to prove the same conclusion when R is a regular ring and $\mathbf{C} \subseteq R$.

Remark 1.13.115. Recall **Theorem 1.4.50 [Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede]**: If R is a regular ring and $\mathfrak{a}$ is a radical ideal with bight $\mathfrak{a} = h$, then for all n, $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.

*Proof of **Theorem 1.4.50 [Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede]**.* Let

$$z \in \mathfrak{a}^{(hn)}$$

and write $q = an + r$ with $0 \leq r \leq n-1$ for some a via the division algorithm. Therefore, $z^a \in \mathfrak{a}^{(han)}$. Also,

$$\mathfrak{a}^{hn} z^a \subseteq \mathfrak{a}^{hn} z^a \subseteq \mathfrak{a}^{(han+hr)} = \mathfrak{a}^{(h(an+r))} = \mathfrak{a}^{(hq)} \subseteq \mathfrak{a}^{[q]},$$

where $\mathfrak{a}^{(hq)} \subseteq \mathfrak{a}^{[q]}$ by **Theorem 1.4.53 [Hochster-Huneke]**. Now, take n^{th} ordinary powers to see that

$$\mathfrak{a}^{hn^2} z^{an} \subseteq \left(\mathfrak{a}^{[q]}\right)^n = (\mathfrak{a}^n)^{[q]}.$$

As $q \geq an$, any $c \in \mathfrak{a}^{hn^2}$ satisfies $cz^q \in (\mathfrak{a}^n)^{[q]}$. Therefore, $z \in (\mathfrak{a}^n)^* = \mathfrak{a}^n$, as desired. □

Remark 1.13.116. Our next goal is to use tight closure to study $\tau(R)$. Note that to establish $z \in \mathfrak{a}^*$, we need to find a c such that $cz^q \in \mathfrak{a}^{[q]}$, but note that c could depend on $\mathfrak{a}$. We might hope for a universal c that works for all $\mathfrak{a}$. That is, a c such that

$$c \in \bigcap_{\mathfrak{a} \subseteq R} (\mathfrak{a} : \mathfrak{a}^*) .$$

There is a priori no reason to except such a c to exist. However, in **Theorem 1.13.41**, we showed that for any domain R, there are elements $c \in R \setminus \{0\}$ such that R_c is regular and $c \in \tau(R, \varphi)$ for fixed $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$. From this, one can show a critical observation:

For any $d \neq 0$, there exists a map φ so that $\varphi(F^e_* d) = c$.

Accepting this, we can see that

$$\bigcap_{\mathfrak{a} \subseteq R} (\mathfrak{a} : \mathfrak{a}^*) \neq \emptyset,$$

since for any $z \in \mathfrak{a}^*$, $dz^{q'} \in \mathfrak{a}^{[q']}$ for $q' \gg 0$ (note d depends on $\mathfrak{a}$), and we can then pick $q = p^e$ and $\varphi : F^e_* R \to R$ such that $\varphi(F^e_* d) = c$. Notice then that

$$cz^{q'} = \varphi\left(F^e_* dz^{qq'}\right) \in \varphi\left(F^e_* \mathfrak{a}^{[qq']}\right) \subseteq \mathfrak{a}^{[q']},$$

where c does **not** depend on $\mathfrak{a}$!

Remark 1.13.117. We will use this to expand our setting to tight closure of modules. To do so, we need a module replacement for $\mathfrak{a}^{[q]}$. Consider the map

$$\gamma_e : M \cong M \otimes_R R \xrightarrow{\mathrm{id} \otimes F^e} M \otimes_R F^e_* R.$$

Definition 1.13.118 (Frobenius power of a module)**.** Denote for $z \in M$, $z^q = \gamma_e(z)$ for $q = p^e$. For $N \subseteq M$, denote $N^{[q]}$ for $\gamma_e(N)$.

☡ **Warning! 1.13.119.** Observe that $N^{[q]} \subseteq M \otimes_R F^e_* R$; that is, $N^{[q]}$ depends on how $N \subseteq M$.

Definition 1.13.120 (tight closure of a module)**.** For a fixed inclusion $N \subseteq M$, set

$$N^*_M = \left\{ z \in M \mid \text{there exists } c \in R^\circ \text{ such that } cz^q \in N^{[q]} \right\}.$$

Lemma 1.13.121. *For $N \subseteq M$, $z \in N^*_M$ if and only if $\overline{z} \in 0^*_{M/N}$.*

Proof. Note that $z \in N^*_M$ if and only if there exists $c \neq 0$ such that $z \otimes c \in \operatorname{im}(N \otimes F^e_* R \to M \otimes F^e_* R)$. Use the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

and the fact that $- \otimes_R F^e_* R$ is right exact. That is,

$$N \otimes_R F^e_* R \to M \otimes_R F^e_* R \to M/N \otimes_R F^e_* R \to 0$$

is exact. □

Remark 1.13.122. Recall while discussing F-rational singularities, in **Definition 1.8.21**, we defined the notation $0^*_{H^d_{\mathfrak{m}}(R)}$, which was no accident! Our previous definition will agree with tight closure.

Definition 1.13.123 ((finitistic) test element)**.** For $c \in R^\circ$, call c a **(finitistic) test element** if for any inclusion $N \subseteq M$ of finitely generated modules, $z \in N^*_M$ if and only if $cz^q \in N^{[q]}$ for $q \gg 0$.

Definition 1.13.124 (big test element)**.** For $c \in R^\circ$, call c a **big test element** if for any inclusion $N \subseteq M$ of modules, $z \in N_M^*$ if and only if $cz^q \in N^{[q]}$ for $q \gg 0$.

Definition 1.13.125 (finitistic test ideal)**.** Define $\tau_{fg}(R)$ to be the ideal generated by all finitistic test elements. That is,

$$\tau_{fg}(R) = \bigcap_{\substack{N \subseteq M \\ \text{finitely generated}}} (N : N_M^*)$$

Definition 1.13.126 (big test ideal)**.** Define $\tau_b(R)$ to be the ideal generated by all big test elements. That is,

$$\tau_{fg}(R) = \bigcap_{N \subseteq M} (N : N_M^*)$$

Remark 1.13.127. It's obvious that $\tau_b(R) \subseteq \tau_{fg}(R)$. Also, $\tau_{fg}(R) = R$ if and only if R is weakly F-regular. Therefore, note that

$$\begin{array}{ccc} \text{strongly } F\text{-regular} & \Longleftrightarrow & \tau(R) = R \\ \Downarrow & & \\ \text{weakly } F\text{-regular} & \Longleftrightarrow & \tau_{fg}(R) = R \end{array}$$

It is conjectured that $\tau_{fg}(R) = \tau_b(R)$. This is called the big = small conjecture, and is equivalent to the weak = strong conjecture.

Theorem 1.13.128. *Big test elements exist.*

Proof. Note that $z \in N_M^*$ if and only if there exists $c \in R^\circ$ such that $cz^q \in N^{[q]}$, but c could depend on N. First, choose c such that R_c is regular. For $d \in R^\circ$, pick $\varphi \in \operatorname{Hom}_R(F_*^{e'} R, R)$ such that $\varphi(F_*^{e'} d) = c$ (by **Remark 1.13.116**). Set $N \subseteq M$ and $z \in N_M^*$; i.e., for $e \gg 0$, there is $d \in R^\circ$ such that $dz^{p^{e+e'}} \in N^{[p^{e+e'}]}$. View φ as a map $F_*^{e+e'} R \to F_*^e R$. Consider the diagram

$$\begin{array}{ccc} N^{[p^{e+e'}]} & \xrightarrow{\varphi} & N^{[p^e]} \\ \big\downarrow & & \big\downarrow \\ M \otimes_R F_*^{e+e'} R & \xrightarrow[\varphi]{} & M \otimes_R F_*^e R. \end{array}$$

Apply φ to $dz^{p^{e+e'}}$ to see that

$$cz^{p^e} = \varphi\left(F_*^{e'} d\right) z^{p^e} = \varphi\left(dz^{p^{e+e'}}\right) \in N^{[p^e]}.$$

Thus, c is a big test element. $\square$

Remark 1.13.129. Our goal is to show that $\tau(R)$ exists. We will do so by showing it is equivalent to $\tau_b(R)$ via an intermediate test ideal, using Matlis dualilty. Henceforth for simplification, assume that $(R, \mathfrak{m}, k)$ is an excellent complete local domain.

Definition 1.13.130 (test ideal 3)**.** Let $(R, \mathfrak{m}, k)$ be an excellent complete local domain. Set $\widetilde{\tau}(R) = \operatorname{Ann} 0_E^*$, where $E = E_R(k)$.

Theorem 1.13.131 (Lyubeznik-Smith-Takagi)**.** $\widetilde{\tau}(R) = \tau_b(R)$.

Proof. The inclusion $\tau_b(R) \subseteq \widetilde{\tau}(R)$ is clear, since

$$\tau_b(R) \subseteq (0 : 0_E^*) = \operatorname{Ann} 0_E^* = \widetilde{\tau}(R).$$

Let $c \in \operatorname{Ann} 0_E^* = \widetilde{\tau}(R)$; for each $d \in \tau_b(R)$, if $z \in 0_E^*$, i.e., $dz^q = 0$, then $cz = 0$, so $c(dz^q) = 0$. That is, $c \in \ker(dF^e-)$ for all $e \gg 0$, where $F^e : E \to F_*^e E$ is the natural Frobenius operator. That is,

$$c \in \bigcap_e \ker\left(dF^e-\right).$$

As $\ker(dF^e-)$ is a descending family of submodules in E and E is artinian, it stabilizes. One can pick a single map $\psi : E \to F_*^e E$ for which

$$c \in \ker d\psi = \bigcap_e \ker(dF^e-).$$

Now $d\psi : E \to F_*^e E$ has a Matlis dual $\varphi : F_*^e R \to R$ with $\varphi(F_*^e d) = c$. That is, we can replicate the proof of **Theorem 1.13.128** to get $c \in \tau_b(R)$. □

Remark 1.13.132. It only remains to connect $\widetilde{\tau}(R)$ to $\tau(R)$. Observe that for each $f \in \mathfrak{a}$, given the natural map

$$R \to F_*^e R \xrightarrow{\cdot F_*^e f} F_*^e R,$$

one has, after taking $\operatorname{Hom}_R(-, R)$,

$$\operatorname{Hom}_R\left(F_*^e R, R\right) \xrightarrow{\operatorname{Hom}_R(-,\cdot F_*^e f)} \operatorname{Hom}_R\left(F_*^e R, R\right) \xrightarrow{ev} \operatorname{Hom}_R(R, R) \cong R \twoheadrightarrow R/\mathfrak{a}.$$

The ideal $\mathfrak{a}$ is compatible if and only if the above map is the zero map.

Lemma 1.13.133 (Schwede)**.** *An ideal $\mathfrak{a}$ is compatible if and only if*

$$\operatorname{Hom}_R\left(F_*^e R, R\right) \xrightarrow{\operatorname{Hom}_R(-,\cdot F_*^e f)} \operatorname{Hom}_R\left(F_*^e R, R\right) \xrightarrow{ev} \operatorname{Hom}_R(R, R) \cong R \twoheadrightarrow R/\mathfrak{a}$$

is zero, and by Matlis duality, $\mathfrak{a}$ is compatible if and only if

$$E_{R/\mathfrak{a}} \to E_R \to E_R \otimes F_*^e R \xrightarrow{\operatorname{Hom}_R(-,\cdot F_*^e f)^\vee} E_R \otimes F_*^e R$$

is zero.

Remark 1.13.134. There is also a fact due to Matlis duality: there is a bijective correspondence

$$\{\text{submodules } N \subseteq E\} \cong \{\text{ideals } J \subseteq R\}$$

given by $N \mapsto \operatorname{Ann} N$ and $J \mapsto E_{R/J}$.

Theorem 1.13.135. $\tau_b(R) \cong \tau(R)$. *Test ideals exist!*

Proof. To see $\tau_b(R)$ is compatible, it's enough to check that

$$E_{R/\tau_b(R)} \to E_R \to E_R \otimes_R F_*^e R \to E_R \otimes_R F_*^e R$$

is zero. Notice that

$$0_E^* \cong E_{R/\operatorname{Ann} 0_E^*} = E_{R/\widetilde{\tau}(R)} \cong E_{R/\tau_b(R)}.$$

If $\mathfrak{a}$ is any compatible ideal, then

$$E_{R/\mathfrak{a}} \to E_R \to E_R \otimes_R F_*^e R \to E_R \otimes_R F_*^e R$$

is zero, but $E_{R/\mathfrak{a}} \subseteq 0_E^*$; i.e.,

$$\mathfrak{a} = \operatorname{Ann} E_{R/\mathfrak{a}} \supseteq \operatorname{Ann} 0_E^* = \widetilde{\tau}(R) \cong \tau_b(R),$$

as we needed to show. □

1.13.2 Frobenius Closure

Remark 1.13.136. We can characterize (weakly) F-regular rings in terms of a closure; R is weakly F-regular if and only if $\mathfrak{a} = \mathfrak{a}^*$ for all ideals $\mathfrak{a} \subseteq R$. We would like to characterize other singularities in terms of closures.

Definition 1.13.137 (Frobenius closure)**.** For an ideal $\mathfrak{a} \subseteq R$, the **Frobenius closure** of $\mathfrak{a}$ is

$$\mathfrak{a}^F = \left\{ z \in R \mid z^q \in \mathfrak{a}^{[q]} \text{ for all } q \gg 0 \right\}.$$

Remark 1.13.138. It's obvious that $\mathfrak{a} \subseteq \mathfrak{a}^F \subseteq \mathfrak{a}^*$. Hence, in a (weakly) F-regular ring, $\mathfrak{a} = \mathfrak{a}^F$ for all $\mathfrak{a} \subseteq R$.

Example 1.13.139. There are ideals for which $\mathfrak{a} \neq \mathfrak{a}^F$. Let $R = {}^{k[u,v,y,z]}\!\big/\!_{(uv, uz, z(v-y^2))}$. One can check that $y^3z^4 \notin (y^2(u^2-z^4))$, but $(y^3z^4)^p \in (y^2(u^2-z^4))^{[p]}$. Thus $(y^2(u^2-z^4))$ is not Frobenius closed. Remark also that R is not F-split.

Lemma 1.13.140. *The operation $\mathfrak{a} \mapsto \mathfrak{a}^F$ is a closure.*

Proof. We only show that if $\mathfrak{a} \subseteq R$ is an ideal, then $\mathfrak{a}^{FF} = \mathfrak{a}^F$. If $z \in \mathfrak{a}^{FF}$, then $z^{p^e} \in (\mathfrak{a}^F)^{[p^e]}$. Write $\mathfrak{a}^F = (x_1, ..., x_n)$; thus

$$z^{p^e} = \sum a_i {x_i}^{p^e}.$$

Each x_i is in $\mathfrak{a}^F$, so ${x_i}^{p^e} \in \mathfrak{a}^{[p^e]}$. Pick e' such that ${x_i}^{p^{e'}} \in \mathfrak{a}^{[p^{e'}]}$ for all i. Thus,

$$z^{p^{e+e'}} = \sum {a_i}^{p^{e'}} {x_i}^{p^{e+e'}} \in \mathfrak{a}^{[p^{e+e'}]},$$

and so $z \in \mathfrak{a}^F$. □

Remark 1.13.141. Our goal is to show that R is F-split if and only if $\mathfrak{a} = \mathfrak{a}^F$ for all $\mathfrak{a} \subseteq R$. We do so in the following steps.

Lemma 1.13.142. *If R is F-split, then $\mathfrak{a} = \mathfrak{a}^F$ for all $\mathfrak{a}$.*

Proof. If $R \to F^e_* R \xrightarrow{\varphi} R$ is a splitting and $z \in \mathfrak{a}^F$ for some $\mathfrak{a} \subseteq R$, then $z^{p^e} \in \mathfrak{a}^{[p^e]}$, so we have $F^e_* z^{p^e} \in F^e_* \mathfrak{a}^{[p^e]}$. Therefore

$$z = \varphi\left(F^e_* z^{p^e}\right) \in \varphi\left(F^e_* \mathfrak{a}^{[p^e]}\right) = \varphi\left(\mathfrak{a} F^e_* R\right) = \mathfrak{a}\varphi\left(F^e_* R\right) = \mathfrak{a}.$$

□

Remark 1.13.143. Notice that if we set $S = F^e_* R$ and view $R \hookrightarrow S$ as a module finite (since rings are F-finite) extension, then for $e \gg 0$, we have

$$\mathfrak{a}S \cap R = \mathfrak{a}F^e_* R \cap R = F^e_* \mathfrak{a}^{[p^e]} \cap R = \left\{ z \in R \mid F^e_* z^{p^e} \in F^e_* \mathfrak{a}^{[p^e]} \right\} = \left\{ z \in R \mid z^{p^e} \in \mathfrak{a}^{[p^e]} \right\} = \mathfrak{a}^F.$$

We can ask if the property $\mathfrak{a} = \mathfrak{a}S \cap R$ is equivalent to $R \hookrightarrow S$ being split. (One can easily show that $R \hookrightarrow S$ split implies $\mathfrak{a} = \mathfrak{a}S \cap R$ by repeating the above proof.)

Theorem 1.13.144 (Hochster)**.** *For a module finite inclusion $R \hookrightarrow S$ of excellent local rings, $\mathfrak{a}S \cap R = \mathfrak{a}$ for all $\mathfrak{a} \subseteq R$ implies $R \hookrightarrow S$ splits.*

Corollary 1.13.145. *A reduced excellent local ring is F-split if and only if $\mathfrak{a} = \mathfrak{a}^F$ for all $\mathfrak{a} \subseteq R$. As $\mathfrak{a} \subseteq \mathfrak{a}^F \subseteq \mathfrak{a}^*$, it's also clear that F-regular implies F-split via the characterizations in terms of closure discussed.*

Remark 1.13.146. Recall **Theorem 1.7.18 [Singh]**; if $R = {}^{k[A,B,C,D,T]}\!\diagup_{I}$ where I is the 2×2 minors of

$$\begin{bmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{bmatrix}.$$

We saw that if $p > 2$, R is not F-regular for $m - \frac{m}{n} > 2$ and not F-split for $\gcd(m,p) = 1$. To verify these claims, one checks explicitly that $B^n T^{m-1} \notin (A, D)$, but $B^n T^{m-1} \in (A, D)^*$. The trick is to set $q = p^e = 2m\ell + \delta$ for $\ell, \delta \in \mathbf{Z}$ with $\ell(m - \frac{m}{n} - 2) \geq 1$ and $-m + 2 \leq \delta \leq 1$. Thus, $q + m - 1 \geq 2m\ell + 1$ and $q \leq 2m\ell + 1$. One can then check carefully that

$$\left(B^n T^{m-1}\right)^{2m\ell+1} \in \left(A^{2m\ell+1}, D^{2m\ell+1}\right).$$

Remark 1.13.147. We can also interpret F-rational, F-injective, and F-nilpotent singularities in terms of tight/Frobenius closure of local cohomology. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. We know via **Corollary 1.8.11** that R is F-rational if and only if $0^*_{H^d_\mathfrak{m}(R)} = 0$ (and note that $H^d_\mathfrak{m}(R) \neq 0$).

Definition 1.13.148 (Frobenius closure of a module)**.** In a similar way to tight closure (**Definition 1.13.120 [tight closure of a module]**), we can extend Frobenius closure to modules, defining

$$0^F_{H^d_\mathfrak{m}(R)} = \left\{ z \in H^d_\mathfrak{m}(R) \mid z^q = 0 \text{ for } q \gg 0 \right\}.$$

Remark 1.13.149. Notice that $0^F_{H^d_\mathfrak{m}(R)} \subseteq 0^*_{H^d_\mathfrak{m}(R)}$. Also, recall that for a non-Cohen-Macaulay local ring $(R, \mathfrak{m})$, R is F-nilpotent if the Frobenius action on $H^i_\mathfrak{m}(R)$ is nilpotent for all $i < d$ and the Frobenius action on $0^*_{H^d_\mathfrak{m}(R)}$ is nilpotent. Immediately, we may recharacterize the second condition as $0^F_{H^d_\mathfrak{m}(R)} = 0^*_{H^d_\mathfrak{m}(R)}$. It is therefore clear from this approach that **Theorem 1.8.15 [Srinivas-Takagi]** holds; F-nilpotent and F-injective implies F-rational.

Remark 1.13.150. Computing $\mathfrak{a}^F$ can be hard, but when $\mathfrak{a}$ is finitely generated, there exists $e \gg 0$ such that $\mathfrak{a}^{[p^e]} = (\mathfrak{a}^F)^{[p^e]}$.

Definition 1.13.151 (Frobenius test exponent)**.** Call $fte(\mathfrak{a})$ the **Frobenius test exponent**. It is the number such that

$$\mathfrak{a}^{\left[p^{fte(\mathfrak{a})}\right]} = \left(\mathfrak{a}^F\right)^{\left[p^{fte(\mathfrak{a})}\right]}.$$

Remark 1.13.152. Recall that in $H^d_\mathfrak{m}(R)$,

$$\cdots \subseteq \ker F^e \subseteq \ker F^{e+1} \subseteq \cdots$$

stabilizes (which is surprising, since $H^d_\mathfrak{m}(R)$ is artinian, not noetherian). The number e' for which $\ker F^{e'} = \ker F^{e'+k}$ for all k is $HSL(H^d_\mathfrak{m}(R))$ (recall **Remark 1.8.47**).

Theorem 1.13.153 (Katzman-Sharp)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. If $\mathfrak{q} \subseteq R$ is an ideal generated by part of a system of parameters for R, then $fte(\mathfrak{q}) \leq HSL(H^d_\mathfrak{m}(R))$.*

Remark 1.13.154. By **Theorem 1.8.48 [Hartshorne-Speiser-Lyubeznik]**, since $H^d_\mathfrak{m}(R)$ is artinian, $HSL(H^d_\mathfrak{m}(R)) < \infty$, and so in the context above, $fte(\mathfrak{q}) < \infty$. Let

$$fte(R) = \sup_{\substack{\mathfrak{q} \text{ generated by parts} \\ \text{of systems of parameters}}} fte(\mathfrak{q}).$$

It is an open question if $fte(R) < \infty$.

Example 1.13.155. Recall **Definition 1.2.6 [system of parameters]**; given a local ring $(R, \mathfrak{m})$ of dimension d, a sequence $x_1, ...x_d \in R$ is a system of parameters if $\sqrt{(x_1, ..., x_d)} = \mathfrak{m}$. For instance, if $R = {}^{k[\![x,y,u,v]\!]}\!\diagup_{(xu - yv)}$, then $x, v, y - u$ is a system of parameters.

Theorem 1.13.156 (Colon Capturing)**.** *Let $(R, \mathfrak{m})$ be a domain. If $x_1, ..., x_d$ is a system of parameters, then*

$$((x_1, ..., x_i) : x_{i+1}) \subseteq (x_1, ..., x_i)^*$$

for all $i \in \{1, ..., d-1\}$.

Corollary 1.13.157. *Let $(R, \mathfrak{m})$ be a domain. If R has a system of parameters $x_1, ..., x_d$ for which $(x_1, ..., x_d)^* = (x_1, ..., x_d)$, then R is Cohen-Macaulay.*

Proof. Recall that R is Cohen-Macaulay if and only if $x_1, ..., x_d$ is a regular sequence (**Definition 1.5.31 [Cohen-Macaulay 2]**). Furthermore, $x_1, ..., x_d$ is a regular sequence if and only if

$$((x_1, ..., x_{i-1}) : x_i) \subseteq (x_1, ..., x_i).$$

By hypothesis, $(x_1, ..., x_i) = (x_1, ..., x_i)^*$, and the result follows by applying **Theorem 1.13.156 [Colon Capturing]**. $\square$

Definition 1.13.158 (parameter ideal)**.** We say that an ideal $\mathfrak{q} \subseteq R$ is a **parameter ideal** if $\mathfrak{q} = (x_1, ..., x_d)$ for some system of parameters $x_1, ..., x_d \in R$.

Remark 1.13.159. One can quickly guess that a ring R is F-rational if and only if all parameter ideals $\mathfrak{q}$ are tightly closed. Indeed, this is the case. Recall that a local ring $(R, \mathfrak{m})$ is F-rational if and only if R is Cohen-Macaulay and $0^*_{H^d_\mathfrak{m}(R)} = 0$, by **Theorem 1.8.10 [Smith]**. Let's prove that theorem, using tools developed since mentioning it.

*Proof of **Theorem 1.8.10 [Smith]**.* We must show that $0^*_{H^d_\mathfrak{m}(R)}$ is the largest proper F-stable submodule of $H^d_\mathfrak{m}(R)$. It's clear that $F(0^*_{H^d_\mathfrak{m}(R)}) \subseteq 0^*_{H^d_\mathfrak{m}(R)}$, since if $z \in 0^*_{H^d_\mathfrak{m}(R)}$, then $cz^{p^e} = 0$, so $c^p(z^p)^{p^e} = 0$, so $z^p = F(z) \in 0^*_{H^d_\mathfrak{m}(R)}$.

If N is proper and stable, then Matlis dualize to see the map

$$\operatorname{Hom}_R\left(H^d_\mathfrak{m}(R)/N, E\right) \to \operatorname{Hom}_R\left(H^d_\mathfrak{m}(R), E\right) \cong \omega_R.$$

As ω_R has rank 1, we can find $c \neq 0$ such that

$$c\omega_R \subseteq \operatorname{Hom}_R\left(H^d_\mathfrak{m}(R)/N, E\right) \subseteq \omega_R.$$

Matlis dualize again to find that

$$\begin{tikzcd} H^d_\mathfrak{m}(R) \arrow[r] \arrow[rr, bend left, "\cdot c"] & H^d_\mathfrak{m}(R)/N \arrow[r] & cH^d_\mathfrak{m}(R) \end{tikzcd}$$

Thus, $cN = 0$. Now, if $z \in N$, then $cF^e(z) \in cF^e(N) \subseteq cN$, as N is F-stable, so $cF^e(z) = 0$. Thus $z \in 0^*_{H^d_\mathfrak{m}(R)}$, and therefore $N \subseteq 0^*_{H^d_\mathfrak{m}(R)}$. $\square$

Remark 1.13.160. Recall from **Remark 1.5.5** that

$$H^d_\mathfrak{m}(R) \cong \varinjlim_t \operatorname{Ext}^d\left(R/\mathfrak{m}^t, R\right) \cong \varinjlim_t R/(x_1{}^t, ..., x_d{}^t)$$

for any system of parameters $x_1, ..., x_d$. A class $\eta \in H^d_\mathfrak{m}(R)$ can be represented under this isomorphism as $\eta = [z + (x_1, ..., x_d)]$, so $F^e(\eta) = [z^{p^e} + (x_1{}^{p^e}, ..., x_d{}^{p^e})]$.

Lemma 1.13.161 (Smith)**.** *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local domain. Let $x_1,...,x_d$ be a system of parameters for R. An element z is in $(x_1,...,x_d)^*$ if and only if $\eta=[z+(x_1,...,x_d)]$ is in $0^*_{H^d_{\mathfrak{m}}(R)}$.*

Proof. If $z\in(x_1,...,x_d)^*$, then $cz^{p^e}\in({x_1}^{p^e},...,{x_d}^{p^e})$, and therefore $cF^e(\eta)=0$.

On the other hand, if $cF^e(\eta)=0$, then $[cz^{p^e}+({x_1}^{p^e},...,{x_d}^{p^e})]=0$. As R is Cohen-Macaulay, $cz^{p^e}\in({x_1}^{p^e},...,{x_d}^{p^e})$, and therefore $z\in(x_1,...,x_d)^*$. □

Theorem 1.13.162 (Smith)**.** *A local domain $(R,\mathfrak{m})$ is F-rational if and only if $\mathfrak{q}=\mathfrak{q}^*$ for all parameter ideals $\mathfrak{q}\subseteq R$.*

Proof. Let $x_1,...,x_d$ be a system of parameters, and let $\mathfrak{q}=(x_1,...,x_d)$ be a parameter ideal. If $(R,\mathfrak{m})$ is F-rational, then for any $z\in\mathfrak{q}^*$, we have $\eta=[z+\mathfrak{q}]\in 0^*_{H^d_{\mathfrak{m}}(R)}=0$, so $z\in\mathfrak{q}$.

Conversely, if $\mathfrak{q}=\mathfrak{q}^*$, then R is Cohen-Macaulay. For any system of parameters $x_1,...,x_d$, an element $\eta=[z+(x_1,...,x_d)]\in 0^*_{H^d_{\mathfrak{m}}(R)}$ must satisfy $cF^e(\eta)=0$, since $\mathfrak{q}=\mathfrak{q}^*$. If $cF^e(\eta)=0$, then $\eta=0$, and therefore $0^*_{H^d_{\mathfrak{m}}(R)}=0$, as desired. □

Remark 1.13.163. First, note that it immediately follows from **Theorem 1.13.162 [Smith]** another way to see that F-regular rings are F-rational.

Note second that the characterization of F-rational rings as those for which $\mathfrak{q}=\mathfrak{q}^*$ for all $\mathfrak{q}\subseteq R$ a parameter ideal is historically the original characterization. **Theorem 1.13.162 [Smith]** showed the equivalence of this original definition and the definition that we presented in **Definition 1.8.1 [F-rational]**.

Third, one might then hope that one can characterize F-injective rings as those for which $\mathfrak{q}=\mathfrak{q}^F$ for all parameter ideals $\mathfrak{q}$. Unfortunately, this is not the case.

Theorem 1.13.164 (Quy-Shimomoto)**.** *If $\mathfrak{q}=\mathfrak{q}^F$ for all parameter ideals $\mathfrak{q}\subseteq R$, then R is F-injective. The converse fails, however.*

Example 1.13.165. To see that the converse fails, let $R=k[\![u,v,y,z,t]\!]\big/(t)\cap(uv,uz,z(v-y^2))$. R is of dimension 4 and F-injective, but R is not F-split, since

$$y^3z^4t\in\left(y^2\left(u^2-z^4\right)\right)^F\setminus\left(y^2\left(u^2-z^4\right)\right).$$

The converse does hold if the length of $H^i_{\mathfrak{m}}(R)$ is finite for $i<\dim R$.

Theorem 1.13.166 (Polstra-Quy)**.** *An equidimensional ring $(R,\mathfrak{m})$ is F-nilpotent if and only if $\mathfrak{q}^F=\mathfrak{q}^*$ for all parameter ideals $\mathfrak{q}\subseteq R$.*

Remark 1.13.167. Finally, let's return to the ongoing diagram one last time (last seen in **Remark 1.10.58**). We have: